RODUCTION TO THE MATHEMATICAL THEORY OF THE CONDUCTION OF HEAT IN SOLIDS

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INTRODUCTION TO THE

MATHEMATICAL THEORY OF THE CONDUCTION OF HEAT IN SOLIDS

BY

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SECOND EDITION, COMPLETELY REVISED

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON

1921

First Published, 1906.

PRINTED IN GREAT BRITAIN

PREFACE

This volume completes the new edition of my book on *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*. The original work was first published in 1906 and has now for some time been out of print. The first volume of the new edition appeared towards the middle of this year, and deals with the Theory of Infinite Series and Integrals, with special reference to Fourier's Series and Integrals. This formed a completely new work with the title *Fourier's Series and Integrals*. The second volume is devoted wholly to the Mathematical Theory of the Conduction of Heat in Solids. This part of the book has also been completely rewritten and much enlarged. It now includes a discussion of all the important boundary problems associated with the Equation of Conduction. The treatment of these questions, especially in the later chapters, should be of use to those interested in the application of modern analysis to the solution of the differential equations of mathematical physics.

In Chapter I. the Differential Equation of Conduction is obtained and some general theorems as to its solution are established. Chapter II. deals with Fourier's Ring. The next two chapters are devoted to Linear Flow. The principal changes made in these chapters are connected with the more exact treatment of the Infinite Series and Integrals which enter into the solutions. Chapters V. and VI., which deal with Two-Dimensional Problems and Flow of Heat in a Rectangular Parallelepiped, differ little from the corresponding chapters in the first edition.

Chapter VII. deals with the Circular Cylinder, Chapter VIII. with the Sphere and Cone, Chapter IX. with Sources and Sinks, and Chapter X. with Green's Functions. These chapters contain much additional matter.

Chapters XI. and XII. are quite new. The former is entitled "The Use of Contour Integrals in the Solution of the Equation of Conduction." Bromwich's recent work has directed attention to the "operational method" of Heaviside. It is true that all the questions examined in this chapter could be solved by that method. But to justify the operational method we must rely upon contour integration, and the chief difference between the method developed by me, as illustrated in this chapter, and the operational method is that I prefer in each case to turn to the standard path in the plane of the complex variable instead of using a kind of mathematical shorthand.

In the last chapter—Chapter XII.—a sketch is given of the use of Integral Equations in the solution of the Equation of Conduction.

This second volume could not have appeared so soon after the first had I not been privileged to spend this year on leave of absence from the University of Sydney in my old College at Cambridge. For the facilities so fully granted to me there I take this opportunity of expressing my heartfelt thanks.

EMMANUEL COLLEGE,
CAMBRIDGE, *October*, 1921.

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[In this volume the author's book *Fourier's Series and Integrals* (2nd Ed.), 1921, will be referred to as *F.S.*]

CHAPTER I

THE DIFFERENTIAL EQUATION OF THE MATHEMATICAL THEORY OF THE CONDUCTION OF HEAT

1. Introductory.

When different parts of a body are at different temperatures, heat flows from the hotter to the colder. Consider the metal rod *ABCD*,

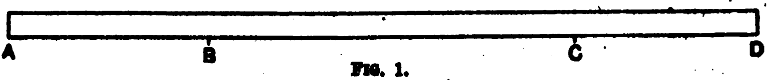

Fig. 1.

and suppose it is heated at the end *A* from some external source. For some time the temperature of the rod gradually rises, the parts near *A* being heated first, but no change takes place at *CD* till *BC* has had its temperature raised. Ultimately, if the end *A* is heated long enough, it is found that a steady state of temperature is reached, in which, while the temperature may vary from point to point, it remains the same at each point as the time changes.

This transference of heat from the hotter portions of a body to the colder is called Conduction of Heat. It must be distinguished from Convection, on the one hand, and Radiation, on the other. In Convection the transference of heat is due to the motion of the heated body itself, as, for example, when the different parts of a liquid are at different temperatures, currents are produced by means of which a uniform temperature is reached. In Radiation the hotter body loses heat and the colder body gains it by means of a process occurring in some intervening medium.

2. Conductivity.

The Mathematical Theory of the Conduction of Heat may be said to be founded upon a hypothesis suggested by the following experiment:

A metal plate is given, bounded by two parallel planes of such an extent that, so far as points well in the centre of the planes are concerned, these bounding surfaces may be supposed infinite. The two planes are kept at different temperatures, the difference not being so great as to cause any sensible change in the properties of the solid. For example, the upper surface may be kept at the temperature of meltiug ice by a supply of pounded ice packed upon it, and the lower at a fixed temperature by having a stream of warm water continually flowing over it. When these conditions have endured for a sufficient time the temperature of the different points of the solid settles down towards its steady value, and at points well removed from the ends the temperature will remain the same along planes parallel to the surfaces of the plate.

Consider the part of the solid bounded by an imaginary cylinder of cross-section S whose axis is normal to the surface of the plate. This cylinder is supposed so far in the centre of the plate that no flow of heat takes place across its generating lines. Let the temperature of the lower surface be v_0° C. and of the upper v_1° C. ($v_0 > v_1$), and let the thickness of the plate be d centimetres. The results of experiments upon different metals suggest that when the steady state of temperature has been reached, the quantity Q of heat which flows up through the plate in t seconds over the surface S is equal to

$$\frac{K(v_0-v_1)St}{d},$$

where K is a constant, called the Thermal Conductivity of the substance, depending upon the material of which it is made. In other words, the flow of heat between these two surfaces is proportional to the difference of temperature of the surfaces.

This result must not be regarded as proved by these experiments. They suggest the law rather than verify it. The more exact verification is to be found in the agreement of experiment with calculations obtained from the mathematical theory based on the assumption of the truth of this law.

Strictly speaking, the conductivity K is not constant for the same substance, but depends upon the temperature. However, when the range of temperature is limited, this change in K may be neglected, and in the ordinary mathematical theory it is assumed that the conductivity does not vary with the temperature. A nearer

approximation to the actual state may be obtained by making K a linear function of the temperature v,

$$e.g.,\ K = K_0(1+av),$$

where a is small.

It is important to notice the dimensions of K, as it is frequently necessary to change the units of length, mass and time in terms of which it is stated.

Since
$$K = \frac{Qd}{(v_0 - v_1)\,St},$$

its dimensions will depend upon those of $Q/(v_0-v_1)$.

The unit of heat is taken as that quantity which will raise unit mass of water 1° C. The dimensions of $Q/(v_0-v_1)$ are then simply $[M]$, since the unit of heat varies jointly as the unit of mass and the value of the degree.

It follows that
$$[K] = \frac{[M]}{[L][T]}.$$

On the C.G.S. system the unit of heat is the *Calory*, the quantity which will raise 1 gramme of water 1° C.*

If it is desired to measure heat by the work necessary to produce it, the dynamical unit in this system would be the *erg*. The relation between the calory and this unit is given to a sufficient approximation by the equation

$$1 \text{ calory} = 4{\cdot}2 \times 10^7 \text{ ergs},$$

and the numerical value of K, when heat is measured in calories, will be $4{\cdot}2 \times 10^7$ times its value when this dynamical unit is employed.†

---

* Another unit sometimes used is the *British Thermal Unit* (B.T.U.), *i.e.* the quantity required to raise 1 pound of water at its maximum density (39° F.) by 1° F.

$$1 \text{ B.T.U.} = 252{\cdot}0 \text{ cal.}$$

† Experiments show that the amount of heat required to raise 1 gramme of water 1° are not quite the same at different temperatures, and in an exact definition of the calory the temperature of the water would need to be specified. It is usual to take for this specified temperature 15° C., and the calory will then be the quantity of heat required to raise 1 gramme of water from 15° C. to 16° C. For this 15° calory we have the equation

$$1 \text{ calory} = 4{\cdot}184 \times 10^7 \text{ ergs.}$$

See Kaye and Laby, *Tables of Physical and Chemical Constants* (4th Ed.), p. 5.

In the fundamental experiment from which our definition of the conductivity is derived, the solid is supposed to be homogeneous and of such a material that, when a point within it is heated, the heat spreads out equally well in all directions. Such a solid is said to be isotropic, as opposed to crystalline and non-isotropic solids, in which certain directions are more favourable for the conduction of heat than others. There are also heterogeneous solids, in which the conditions of conduction vary from point to point as well as in direction at each point. In this book we shall examine only the Theory of Conduction in Homogeneous Isotropic Solids.

3. The Flow of Heat across an Isothermal Surface.

Consider an isotropic solid with a distribution of temperature at the time t given by

$$v=f(x, y, z, t).$$

We may suppose a surface described in the solid, such that at every point upon it the temperature at this instant is the same, say V°. Such a surface is called the Isothermal Surface for the temperature V°, and it may be looked upon as separating the parts of the body which are hotter than V° from the parts which are cooler than V°. We may imagine the isothermals drawn for this instant for different degrees and fractions of a degree. These surfaces may be formed in any way, but no two isothermals can cut each other, since no part of the body can have two temperatures at the same time. The solid is thus pictured as divided up into thin shells by its isothermals. Heat is flowing from one shell to another, this flow of heat being along the normals to the surfaces, as no transference of heat takes place along the surfaces of equal temperature.

Generalising the result of § 2 we take *as our fundamental hypothesis for the Mathematical Theory of the Conduction of Heat that the rate at which heat crosses from the inside to the outside of an isothermal surface per unit area per unit time is equal to*

$$-K\frac{\partial v}{\partial n},$$

where v *is the temperature of the surface,* K *the Thermal Conductivity of the substance, and* $\frac{\partial}{\partial n}$ *denotes differentiation along the outward-drawn normal to the surface.*

As a particular case, when the isothermals are planes perpendicular to the axis of x, the rate of flow of heat per unit area per unit time is $-K\dfrac{\partial v}{\partial x}$ in the direction of the positive axis of x. If v is decreasing as x increases, this rate will be positive. If v increases as x increases, the rate will be negative, meaning that the flow of heat is in the direction of the negative axis of x.

4. The Flow of Heat across any Surface.

We have stated in the preceding article that we assume that the rate of flow of heat across an isothermal at a point P is

$$-K\frac{\partial v}{\partial n}$$

per unit area per unit time, or, in the language of differentials,

$$dQ=-K\frac{\partial v}{\partial n}dS\,dt,$$

dS being an element of the isothermal surrounding the point P. We proceed to obtain an analogous expression for the rate at which heat flows across any surface, not necessarily isothermal, per unit area per unit time at any point P.

We shall denote this rate of flow by f. The value of f will depend upon the position of the point, the direction of the normal to the surface at that point, and the time. We shall now show that, if the values of f are given for three mutually perpendicular planes meeting at a point, its value for any other plane through the point may be written down.

Consider the elementary tetrahedron $PABC$, whose three faces PBC, PCA, PAB are parallel to the coordinate planes, while the perpendicular to the face ABC from the point P has the direction-cosines (λ, μ, ν), and is of length p. (Fig. 2.)

Let the area of ABC be Δ; then the areas of PBC, PCA and PAB are respectively $\lambda\Delta$, $\mu\Delta$ and $\nu\Delta$.

If we denote the rates of flow for the elementary areas PBC, PCA, PAB and ABC by f_x, f_y, f_z, and f, the rate at which heat is gained by the tetrahedron is ultimately given by

$$(\lambda f_x+\mu f_y+\nu f_z-f)\Delta.$$

However, if c and ρ are the specific heat and density, this rate of gain of heat is equal to

$$\frac{1}{3}p\Delta\, c\rho\,\frac{\partial v}{\partial t}.$$

Proceeding to the limit when $p\to 0$, this expression becomes zero, and f_x, f_y, f_z and f become the rates of flow at the point P across

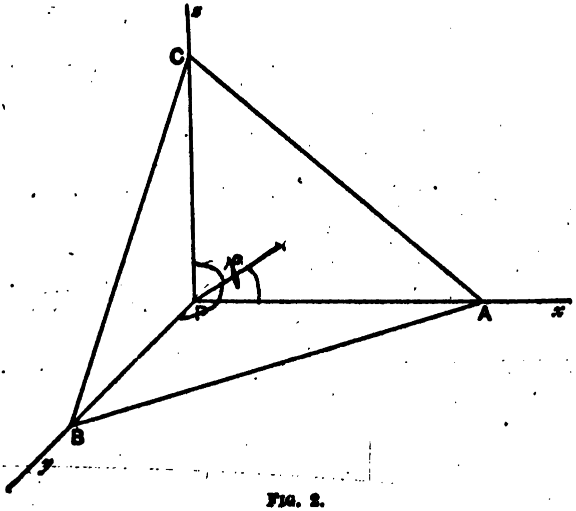

FIG. 2.

planes parallel to the coordinate planes and a plane through P, the perpendicular to which is in the direction (λ, μ, ν). Thus we have

$$\lambda f_x+\mu f_y+\nu f_z=f.$$

Now, according to our fundamental hypothesis, the rate of flow of heat across an isothermal surface per unit area per unit time is equal to the product of the conductivity and the rate of diminution of the temperature in the direction of the normal to the surface. Let P be a point upon the isothermal, and the normal at P the axis of z, the axes of x and y being in the tangent plane through P. Then f_x and f_y are both zero, since no flow takes place along the surface.

Therefore

$$\begin{aligned} f&=\nu f_z\\ &=-K\nu\frac{\partial v}{\partial z}\\ &=-K\frac{\partial v}{\partial h},\end{aligned}$$

where $\frac{\partial}{\partial h}$ denotes differentiation in the direction (λ, μ, ν), since

$$\frac{\partial v}{\partial h}=\lambda\frac{\partial v}{\partial x}+\mu\frac{\partial v}{\partial y}+\nu\frac{\partial v}{\partial z} \quad \text{and} \quad \frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0.$$

Thus *the rate of flow of heat at a point across any surface from the inside to the outside per unit area per unit time is*

$$-K\frac{\partial v}{\partial n},$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward-drawn normal to the surface at the point.

5. The Equation of Conduction.

Consider the case of a homogeneous isotropic solid heated in any way and then allowed to cool. The temperature v at the point $P(x, y, z)$ will be a continuous function of x, y, z and t, and the first differential coefficients of v will also be continuous.

Consider an element of volume of the solid at the point P, namely, the rectangular parallelepiped with this point as centre, its edges being parallel to the coordinate axes, and of lengths $2dx$, $2dy$ and $2dz$.

Let $ABCD$ and $A'B'C'D'$ be the faces to which the axis of x is perpendicular. Then the rate at which heat is flowing into the parallelepiped over the face $ABCD\,(x-dx)$ will ultimately be given by

$$4dy\,dz\left(f_x-\frac{\partial f_x}{\partial x}dx\right),$$

where f_x is the rate of flow at P across the corresponding plane. Similarly the rate at which heat is flowing out across the face $A'B'C'D'$ is given by

$$4dy\,dz\left(f_x+\frac{\partial f_x}{\partial x}dx\right).$$

Thus the rate of gain of heat from these two faces is equal to

$$-8dx\,dy\,dz\frac{\partial f_x}{\partial x}.$$

Similarly from the others we obtain

$$-8dx\,dy\,dz\frac{\partial f_y}{\partial y} \quad \text{and} \quad -8dx\,dy\,dz\frac{\partial f_z}{\partial z}.$$

But this element of volume is gaining heat at the rate of

$$8dx\,dy\,dz\,c\rho\frac{\partial v}{\partial t}.$$

Therefore we have

$$\frac{\partial v}{\partial t}+\frac{1}{c\rho}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)=0.$$

But we have seen that

$$f_x=-K\frac{\partial v}{\partial x},\quad f_y=-K\frac{\partial v}{\partial y},\quad f_z=-K\frac{\partial v}{\partial z}.$$

and K is independent of x, y and z.

Therefore our equation becomes

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

where

$$\kappa=\frac{K}{c\rho}.$$

The constant κ was called by Kelvin the Diffusivity of the substance, and by Clerk-Maxwell its Thermometric Conductivity.

The dimensions of the diffusivity κ are obtained at once from those of the conductivity K (cf. p. 3). Since c above is the ratio of the quantity of heat required to raise unit mass of the substance 1° C. to the quantity required to raise unit mass of water 1° C., it is of zero dimensions in mass, length and time. Also the dimensions of the density ρ are $[M]/[L^3]$.

Thus we have

$$[\kappa]=\frac{[L^2]}{[T]}.$$

It follows that if the units of length and time are the foot and year instead of the centimetre and second, the value of κ for these units will have to be multiplied by $(30{\cdot}48)^2/3{\cdot}1557\times10^7$ to reduce it to the C.G.S. system. (Cf. p. 58.)

By some of the early writers who did not employ the C.G.S. system, the thermal unit was taken as the amount of heat which would raise unit *volume* of water 1° C.

Let c of these units be required to raise unit volume of the substance 1° C.

Then the equation of conduction takes the form

$$\frac{\partial v}{\partial t}=\frac{K}{c}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

where K is the conductivity in terms of the new unit.

It is clear that K/c in this system is equal to the diffusivity $K/\rho c$ discussed above.

On the other hand, when the thermal unit is the amount of heat required to raise unit *volume* of water 1° C., the numerical value of the conductivity will not agree with that obtained when the unit is the amount required to raise unit *mass* of water 1° C., unless the linear unit is the centimetre.

If the solid is isotropic, but not homogeneous, the equation for v becomes

$$c\rho\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right).$$

In the case of Steady Temperature, when the temperature does not vary with the time, the equation becomes that of Potential. Also if at the point $P(x, y, z)$ there exists a source of heat supplying in the time dt the quantity $A\,dt$ of heat per unit volume, the equation becomes

$$c\rho\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right)+A.$$

Such a condition is realised when conduction takes place along a wire along which an electric current is flowing, since this current is generating heat in accordance with Joule's Law.

These results may also be obtained by the application of Green's Theorem,* that when ξ, η, ζ, as well as their first differential coefficients, are continuous functions of x, y and z, inside a closed surface,

$$\iint(l\xi+m\eta+n\zeta)\,dS=\iiint\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)dx\,dy\,dz,$$

(l, m, n) being the direction-cosines of the outward-drawn normal, and the integrations being taken over the surface, and throughout its volume.

Suppose any such surface drawn lying wholly inside the given solid.

The rate at which heat flows out across the element dS of the surface is

$$(lf_x+mf_y+nf_z)\,dS.$$

Therefore the total rate of gain of heat within the surface is

$$-\iint(lf_x+mf_y+nf_z)\,dS.$$

But this rate of gain of heat may also be expressed by

$$\iiint\left(c\rho\frac{\partial v}{\partial t}\right)dx\,dy\,dz,$$

the integration being taken through the region bounded by this surface.

Thus
$$\iiint c\rho\frac{\partial v}{\partial t}dx\,dy\,dz+\iint(lf_x+mf_y+nf_z)\,dS=0.$$

Therefore by Green's Theorem

$$\iiint\left(c\rho\frac{\partial v}{\partial t}+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)dx\,dy\,dz=0,$$

and this holds whatever closed surface we consider, provided it lies wholly within the solid and no source of heat exists within it.

* Cf. Lamb, *Hydrodynamics*, § 42.

Apply this result to the element surrounding the point $P(x, y, z)$, and we obtain the equation

$$c\rho\frac{\partial v}{\partial t}+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}=0,$$

as before.

6. The Transformation of Coordinates.

These equations may be easily transformed into other systems of orthogonal coordinates, the most useful being the Spherical Polar System, in which the position of the point is determined by its distance r from the origin, its latitude θ, and its azimuth ϕ, and the Cylindrical System, in which its position is determined by the polar coordinates r, θ of its projection on the plane of x, y, and the coordinate z.

These are special cases of the general system of orthogonal coordinates, in which the position of a point is given by the intersection of the three orthogonal surfaces,

$$\xi=\text{const.},\quad \eta=\text{const.},\quad \zeta=\text{const.}$$

We proceed to show how this transformation may most easily be affected.

Consider the element of volume bounded by the surfaces $\xi\pm d\xi$, $\eta\pm d\eta$, $\zeta\pm d\zeta$, and let $A'B'C'D'$ and $ABCD$ be the faces $\xi\pm d\xi$.

Let $$ds^2=\lambda^2 d\xi^2+\mu^2 d\eta^2+\nu^2 d\zeta^2$$

be the equation giving the length of the elementary arc joining the points (ξ, η, ζ) and $(\xi+d\xi, \eta+d\eta, \zeta+d\zeta)$.

Then the area of the section of the ξ surface through $P(\xi, \eta, \zeta)$ cut off by the surfaces $\eta\pm d\eta$, $\zeta\pm d\zeta$ is given by

$$4\mu\nu\, d\eta\, d\zeta,$$

and the rate at which heat flows across this section per unit time is

$$4\mu\nu\, d\eta\, d\zeta f_\xi,$$

f_ξ being the rate of flow of heat at P across the surface ξ.

Therefore the rate at which heat flows into the element across the face $ABCD$ is ultimately

$$4\left(\mu\nu f_\xi-\frac{\partial}{\partial\xi}(\mu\nu f_\xi)\, d\xi\right) d\eta\, d\zeta,$$

and the rate at which heat flows out across the face $A'B'C'D'$ is

$$4\left(\mu\nu f_\xi+\frac{\partial}{\partial\xi}(\mu\nu f_\xi)\, d\xi\right) d\eta\, d\zeta.$$

Hence the total rate of gain of heat from these two faces is

$$-8\frac{\partial}{\partial\xi}(\mu\nu f_\xi)\,d\xi\,d\eta\,d\zeta.$$

The other faces give respectively

$$-8\frac{\partial}{\partial\eta}(\nu\lambda f_\eta)\,d\xi\,d\eta\,d\zeta,\qquad -8\frac{\partial}{\partial\zeta}(\lambda\mu f_\zeta)\,d\xi\,d\eta\,d\zeta.$$

Inserting the values of f_ξ, f_η and f_ζ, namely,

$$f_\xi=-\frac{K}{\lambda}\frac{\partial v}{\partial\xi},\quad f_\eta=-\frac{K}{\mu}\frac{\partial v}{\partial\eta},\quad f_\zeta=-\frac{K}{\nu}\frac{\partial v}{\partial\zeta},$$

and equating the expression we thus obtain to

$$8\lambda\mu\nu\,d\xi\,d\eta\,d\zeta\,c\rho\frac{\partial v}{\partial t},$$

we have

$$\lambda\mu\nu c\rho\frac{\partial v}{\partial t}=\frac{\partial}{\partial\xi}\left(\frac{\mu\nu}{\lambda}K\frac{\partial v}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{\nu\lambda}{\mu}K\frac{\partial v}{\partial\eta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{\lambda\mu}{\nu}K\frac{\partial v}{\partial\zeta}\right),$$

which reduces to

$$\lambda\mu\nu\frac{\partial v}{\partial t}=\kappa\left[\frac{\partial}{\partial\xi}\left(\frac{\mu\nu}{\lambda}\frac{\partial v}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{\nu\lambda}{\mu}\frac{\partial v}{\partial\eta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{\lambda\mu}{\nu}\frac{\partial v}{\partial\zeta}\right)\right],$$

when K is constant, and as usual we have written $\kappa=\dfrac{K}{c\rho}$.

Spherical Polar Coordinates.

In this system $x=r\sin\theta\cos\phi$,

$y=r\sin\theta\sin\phi$,

$z=r\cos\theta$,

and $ds^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2$.

Therefore the equation for v becomes

$$\frac{\partial v}{\partial t}=\frac{\kappa}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}\right],$$

which may be written

$$\frac{\partial v}{\partial t}=\kappa\left[\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial v}{\partial\mu}\right)+\frac{1}{r^2(1-\mu^2)}\frac{\partial^2 v}{\partial\phi^2}\right],$$

where $\mu=\cos\theta$.

Cylindrical Coordinates.

In this system $x=r\cos\theta,\quad y=r\sin\theta$,

and $ds^2=dr^2+r^2d\theta^2+dz^2$.

Therefore the equation for v becomes

$$\frac{\partial v}{\partial t}=\frac{\kappa}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial v}{\partial z}\right)\right],$$

which may be written

$$\frac{\partial v}{\partial t}=\kappa\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}\right].$$

7. Initial and Boundary Conditions.

Before we can proceed to the mathematical discussion of the problems of Conduction, it is necessary to determine the formulae which will express the Initial and Boundary Conditions which the temperature satisfies. These are partly the direct expression of the results of experiment and partly the mathematical statement of hypotheses founded upon these results.

I. We assume that in the interior of the solid v is a continuous function of x, y, z and t; and that this holds also of the first differential coefficient with regard to t and of the first and second differential coefficients with regard to x, y and z.

II. *Initial Conditions.*

The temperature throughout the body is supposed given arbitrarily at the instant which we take as the origin of the time coordinate t. If this arbitrary function is continuous, we require to find a solution of our problem which shall, as t converges to zero, also converge to this value. In other words, if the initial temperature is given by

$$v=f(x, y, z),$$

our solution of the equation,

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v,$$

must be such that $$\operatorname*{Lt}_{t\to 0}(v)=f(x, y, z)$$

at all points of the solid.

If the initial distribution is discontinuous at points or surfaces, these discontinuities must disappear after ever so short a time, and in this case our solution must converge to the value given by the initial temperature at all points where this distribution is continuous.

III. *Boundary or Surface Conditions.*

(A) *The Surface of Separation of two Media of Different Conductivities* K_1 *and* K_2.

Let v_1 and v_2 denote the temperatures in the two media. Then it is assumed that at the surface of separation the temperatures are the same.

Suppose an element of area dS taken upon the surface of separation, and that an element of volume is constructed by measuring off lengths ϵ along the normals over this area into both media, the quantity ϵ being an infinitesimal of a lower order than the linear dimensions of dS.

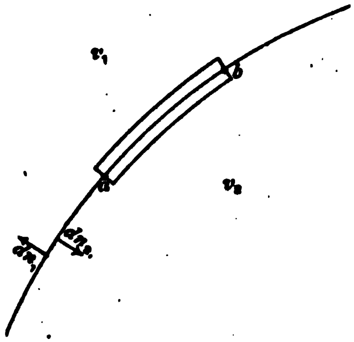

FIG. 2.

Then the rate at which heat is gained by this element of volume from the flow over the surface will ultimately be

$$\left(K_1 \frac{\partial v_1}{\partial n_1} + K_2 \frac{\partial v_2}{\partial n_2}\right) dS,$$

the differentiations being taken along the normals from the common surface into each medium, the contribution from the ends being negligible.

Equating this to the expression

$$2\epsilon\left(c_1\rho_1 \frac{\partial v_1}{\partial t} + c_2\rho_2 \frac{\partial v_2}{\partial t}\right) dS,$$

c_1, c_2 being the specific heats and ρ_1, ρ_2 the densities of the two media, and proceeding to the limit when ϵ vanishes, we have

$$K_1 \frac{\partial v_1}{\partial n_1} + K_2 \frac{\partial v_2}{\partial n_2} = 0,$$

and

$$v_1 = v_2,$$

as the conditions at the surface of separation of the two substances.

(B) When radiation takes place at the surface of the solid into a gas at the temperature v_0, it is assumed, and the assumption

is suggested by experiment, that the loss of heat per unit area per unit time is proportional to the difference of the temperatures of the surface and the gas. In other words this loss of heat is $H(v-v_0)$, where H is a constant associated with the state of the solid and its surface. This quantity is called the Emissivity or Exterior Conductivity, and it is found to vary considerably with the temperature and the state of the surface, so that in experiments on conduction it is best, as far as possible, always to reduce the loss of heat by radiation at the surface to the magnitude of a small correction by treating the surface with a suitable material.

The conditions at the surface follow in the same way as above, and we have

$$\frac{\partial v}{\partial n}+h(v-v_0)=0,$$

where $H/K=h$, and the differentiation is taken along the outward-drawn normal.

(C) There are other possible surface conditions. The boundary may be kept at a constant temperature, or at a temperature which varies with the position of the point and with the time; or the boundary may be rendered impervious to heat. The analytical expressions for these cases are obvious.

In the mathematical treatment of the question these surface and initial conditions are not regarded as conditions which v must satisfy on the surface itself or at the instant $t=0$. They are taken as limiting conditions, and it is required in the one case that our solution shall converge to the given surface or initial value, and in the other case that the differential coefficients in the limit as we approach the surface shall satisfy the corresponding conditions.

8. The Solution of the Problem is Unique.

We shall now show that the general problem of conduction, based upon the equation of conduction and these initial and surface conditions, admits of only one solution.

If possible, let there be two independent solutions v_1, v_2 of the equations

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v \text{ in the solid,}$$

$$v=f(x, y, z) \text{ for } t=0 \text{ in the solid,}$$

$$v=\phi(x, y, z, t) \text{ at the surface.}$$

Let $\quad V=v_1-v_2.$

Then V satisfies

$$\frac{\partial V}{\partial t}=\kappa\nabla^2 V \text{ in the solid,}$$

$$V=0 \text{ for } t=0 \text{ in the solid,}$$

$$V=0 \text{ at the surface.}$$

We shall prove that V must be zero everywhere in the solid.

Consider the volume integral

$$J=\iiint \frac{V_t^2}{2}\,dx\,dy\,dz,$$

the integration being taken through the solid.

Then $$\frac{\partial J}{\partial t}=\iiint V\frac{\partial V}{\partial t}\,dx\,dy\,dz$$

$$=\kappa\iiint V\nabla^2 V\,dx\,dy\,dz.$$

But by Green's Theorem,

$$\iint V\frac{\partial V}{\partial n}dS=\iiint V\nabla^2 V\,dx\,dy\,dz$$

$$+\iiint\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}dx\,dy\,dz,$$

the integrals being taken over the surface and through the volume of the solid.

Therefore

$$\frac{\partial J}{\partial t}=\kappa\iint V\frac{\partial V}{\partial n}\,dS-\kappa\iiint\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}dx\,dy\,dz.$$

Since V is zero over the surface, the first integral vanishes, and we obtain

$$\frac{\partial J}{\partial t}=-\kappa\iiint\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}dx\,dy\,dz.$$

Therefore $$\frac{\partial J}{\partial t}\leqq 0.$$

Since $J=0$ when $t=0$, it follows that

$$J\leqq 0.$$

But since $$J=\iiint\frac{V^2}{2}\,dx\,dy\,dz,$$

$$J\geqq 0.$$

Thus we must have $J=0$ and $V=0$.

A similar discussion shows that there can be only one solution for the problem with the other Boundary Conditions and for the case of Steady Temperature. To prove that the equations must have a solution is another matter. Their physical interpretation requires that this be true: the mathematical demonstration of such Existence Theorems belongs to Pure Analysis.

9. Simplification of the General Problem of Conduction.

When the surface conditions do not vary with the time, we may reduce the general problem to depend upon two simpler cases, one of these being a case of Steady Temperature.

For example, when we have to satisfy

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v, \text{ through the solid,}$$

$$v=f(x, y, z) \text{ initially,}$$

and $$v=\phi(x, y, z) \text{ at the surface,}$$

we may put $$v=u+w,$$

where u is a function of x, y, z only, and satisfies

$$\nabla^2 u=0 \text{ through the solid,}$$

and $$u=\phi(x, y, z) \text{ at the surface;}$$

and w is a function of x, y, z and t, such that

$$\frac{\partial w}{\partial t}=\kappa\nabla^2 w \text{ through the solid,}$$

$$w=f(x, y, z)-u \text{ initially,}$$

and $$w=0 \text{ at the surface.}$$

The first is a case of Steady Temperature, and the second is a case of Variable Temperature with zero surface temperature.

The case of Radiation into a medium whose temperature does not vary with the time may be treated in the same way.

When the surface temperature varies with the time, or when radiation takes place at the surface into a medium whose temperature varies with the time, three different methods may be employed. The first is due to Duhamel, who showed that these two cases could be reduced to those of constant surface temperature or radiation into a medium at constant temperature. The second method corresponds to the use of Green's Function in the Theory

of Potential. (Cf. Ch. X.) The third involves the use of contour integrals. (Cf. Ch. XI.)

At this stage we shall refer only to Duhamel's method, which depends upon the following theorem: *

I. *If $v=F(x, y, z, \lambda, t)$ represents the temperature at (x, y, z) at the time t in a solid in which the initial temperature is zero, while its surface temperature is $\phi(x, y, z, \lambda)$, then the solution of the problem in which the initial temperature is zero, and the surface temperature is $\phi(x, y, z, t)$, is given by*

$$v=\int_0^t \frac{\partial}{\partial t} F(x, y, z, \lambda, t-\lambda)\, d\lambda.$$

When the surface temperature is zero from $t=-\infty$ to $t=0$, and $\phi(x, y, z, \lambda)$ from $t=0$ to $t=t$, we may say that the initial temperature is zero and the surface temperature is $\phi(x, y, z, \lambda)$, so that the temperature at the time t is given by

$$v=F(x, y, z, \lambda, t), \text{ when } t>0.$$

Therefore when the surface temperature is zero from $t=-\infty$ to $t=\lambda$ and $\phi(x, y, z, \lambda)$ from $t=\lambda$ to $t=t$, we have

$$v=F(x, y, z, \lambda, t-\lambda), \text{ when } t>\lambda.$$

Also when the surface temperature is zero from $t=-\infty$ to $t=\lambda+d\lambda$ and $\phi(x, y, z, \lambda)$ from $t=\lambda+d\lambda$ to $t=t$, we have

$$v=F(x, y, z, \lambda, t-\lambda-d\lambda), \text{ when } t>\lambda+d\lambda.$$

Hence when the surface temperature is zero from $t=-\infty$ to $t=\lambda$, $\phi(x, y, z, \lambda)$ from $t=\lambda$ to $t=\lambda+d\lambda$, and zero from $t=\lambda+d\lambda$ to $t=t$, we have

$$v=F(x, y, z, \lambda, t-\lambda)-F(x, y, z, \lambda, t-\lambda-d\lambda),$$

or ultimately

$$v=\frac{\partial}{\partial t} F(x, y, z, \lambda, t-\lambda)\, d\lambda. \quad (t>\lambda)$$

In this way, by breaking up the interval $t=0$ to $t=t$ into these small intervals, and then summing the results thus obtained, we find the solution of the problem for the surface temperature $\phi(x, y, z, t)$ in the form

$$v=\int_0^t \frac{\partial}{\partial t} F(x, y, z, \lambda, t-\lambda)\, d\lambda.$$

* Cf. *J. éc. polytech., Paris*, 14, Cah. 22, p. 20, 1833.

The corresponding theorem for the case of radiation is follows:

II. *If* $v=F(x, y, z, \lambda, t)$ *represents the temperature at* (x, y, z) *the time* t *in a solid in which the initial temperature is zero, wl radiation takes place at its surface into a medium at* $\phi(x, y, z,$ *then the solution of the problem in which the initial temperature zero, and the temperature of the medium is* $\phi(x, y, z, t)$, *is given by*

$$v=\int_0^t \frac{\partial}{\partial t} F(x, y, z, \lambda, t-\lambda)\, d\lambda.$$

When the surface temperature, or the temperature of the mediu into which radiation takes place, does not vary from point to poi but changes only with the time, these results may be stated in slightly simpler form as follows:

III. *If* $v=F(x, y, z, t)$ *represents the temperature at* (x, y, z) *at time* t *in a solid in which the initial temperature is zero, while its surfa is kept at temperature unity* [or, *in the case of radiation, while radiati takes place into a medium at temperature unity*], *then the soluti of the problem when the surface is kept at temperature* $\phi(t)$ [or, *the case of radiation, while radiation takes place into a medium temperature* $\phi(t)$], *is given by*

$$v=\int_0^t \phi(\lambda) \frac{\partial}{\partial t} F(x, y, z, t-\lambda)\, d\lambda.$$

Now the general problem with varying surface temperatu requires the solution of the equations

$$\frac{\partial v}{\partial t}=\kappa \nabla^2 v \text{ through the solid,}$$

$$v=f(x, y, z) \text{ initially,}$$

$$v=\phi(x, y, z, t) \text{ at the surface.}$$

Put $v=u+w$, where

$$\frac{\partial u}{\partial t}=\kappa \nabla^2 u \text{ through the solid,}$$

$$u=0 \text{ initially,}$$

$$u=\phi(x, y, z, t) \text{ at the surface;}$$

and
$$\frac{\partial w}{\partial t}=\kappa\nabla^2 w \text{ through the solid,}$$
$$w=f(x, y, z) \text{ initially,}$$
$$w=0 \text{ at the surface.}$$

The equations for u we have just discussed. Those for w are in their simplest form. Hence Duhamel's Theorem simplifies this problem and reduces it to the case of surface temperature independent of the time.

CHAPTER II

FOURIER'S RING

10. Introductory.

From reasons of symmetry in the initial distribution of temperature or the form of the solid, it will often happen that the equations for the temperature which we have obtained in the previous chapter may be simplified, and that one, and sometimes two, of the coordinates disappear from these equations. For example, if we are dealing with a sphere in which the initial temperature is a function only of the distance r from the centre, and the surface conditions are the same all over the sphere, the temperature will depend only upon r and t. Similarly, if the solid is bounded by two parallel planes, $x=0$ and $x=a$, and if the initial temperature is a function of x only, and the surfaces are kept at constant temperatures, the isothermals will remain planes parallel to the bounding planes, and the temperature will depend only upon x and t. Further, in the case of an infinite cylinder whose generating lines are parallel to the axis of z, when the initial distribution is the same at all points on lines parallel to this axis, and the boundary conditions are of the same nature, the temperature will depend only upon x, y and t, and will be the same at points in the cylinder which lie on lines parallel to the axis.

11. The Equation of Conduction in Fourier's Ring.

One of the simplest and most suggestive problems in the Conduction of Heat, when the temperature depends only upon one coordinate and the time, is Fourier's Problem of the Ring. This problem is also of special interest, as it was the first to which Fourier applied his mathematical theory, and for which the results of his mathematical investigation were compared with the facts of experiment.*

* Fourier, *Théorie analytique de la chaleur*, Ch. II. and IV.

For simplicity we shall suppose the ring to be formed by the revolution of its normal cross-section, a circle of small radius, about an axis perpendicular to the plane of the ring, though the investigation will also apply to any curved bar of small cross-section, the axis of which forms a closed curve with no loops.

The cross-section is supposed so small that the temperature may be regarded as the same at all points of the section. The initial distribution is given, and the problem is to determine the temperature at any point in the ring when it has been allowed to cool by radiation and conduction, or by conduction alone, when the surface is impervious to heat.

We choose the length x from a fixed point on the circle passing through the centres of the normal sections as the coordinate defining the position of a point on this circle. We examine the movement of heat in an element of volume contained between the sections ab and $a'b'$ at distances x and $x+dx$ from the origin, the area of the cross-section being ω and the perimeter p.

The rate at which heat flows into this element over the face ab is equal to

$$-K\frac{\partial v}{\partial x}\omega,$$

and the rate at which it flows out over $a'b'$ is

$$\left(-K\frac{\partial v}{\partial x}-K\frac{\partial^2 v}{\partial x^2}dx-\ldots\right)\omega.$$

Hence ultimately the rate of gain of heat due to the two ends is given by

$$K\frac{\partial^2 v}{\partial x^2}\omega\, dx.$$

The rate at which heat is being lost by radiation at the surface of the element is

$$H(v-v_0)p\,dx,$$

where H is the emissivity: and the total rate of gain of heat is therefore ultimately

$$\left(K\frac{\partial^2 v}{\partial x^2}\omega-pH(v-v_0)\right)dx.$$

But, if c is the specific heat and ρ the density of the substance, this rate of gain of heat is also ultimately equal to

$$c\rho\frac{\partial v}{\partial t}\omega\, dx.$$

Therefore
$$\frac{\partial v}{\partial t}=\frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2}-\frac{Hp}{c\rho\omega}(v-v_0).$$

Writing
$$\frac{Hp}{c\rho\omega}=\lambda \quad\text{and}\quad \frac{K}{c\rho}=\kappa,$$
we have
$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-\lambda(v-v_0).$$

When the surface is rendered impervious to heat H is zero, a[nd] the equation becomes
$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}.$$

Also the case in which there is radiation may be reduced to th[is] form by substituting
$$v=v_0+ue^{-\lambda t},$$
the external temperature v_0 being constant.

It will be noticed that when K is not constant, a similar discussio[n] leads to the equation
$$\frac{\partial v}{\partial t}=\frac{1}{c\rho}\frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x}\right)-\frac{Hp}{c\rho\omega}(v-v_0).$$

12. The Variable Temperature of the Ring.

Consider the distribution of temperature in such a homogeneou[s] isotropic ring of unit radius, when there is no radiation at the surfac[e] and the initial temperature is an arbitrary continuous function $f(x)$ satisfying Dirichlet's Conditions (cf. *F.S.*, § 93),* while $f(-\pi)=f(\pi)$. In this problem we shall suppose this arbitrary function continuou[s]. In the other cases of Linear Flow of Heat the difficulties introduce[d] by discontinuities in the initial temperature will be examined.†

The equations for the temperature are the following:

$$(1)\qquad \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},\quad (t>0,\ -\pi<x<\pi)$$

$$(2)\qquad v=f(x),\quad (t=0,\ -\pi\leqq x\leqq\pi)$$

$$(3)\qquad \left.\begin{aligned} v_{x=\pi}&=v_{x=-\pi}\\ \left(\frac{\partial v}{\partial x}\right)_{x=\pi}&=\left(\frac{\partial v}{\partial x}\right)_{x=-\pi}\end{aligned}\right\},\quad (t>0)$$

the third condition simply expressing the fact that the temperatu[re] and the flow of heat must be continuous at the point given b[y] $x=\pm\pi$, or the opposite end of the diameter through the origin.

Let the Fourier's Series for $f(x)$ be
$$a_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)+\dots,$$

* In this volume the author's book *Fourier's Series and Integrals* (2nd E[d.] 1921, will be referred to as *F.S.*

† Cf. *e.g.* §§ 17, 30, 31.

so that
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx',$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos nx'\,dx',$$
and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\sin nx'\,dx'.$$

Consider the function v defined by the infinite series
$$a_0 + (a_1\cos x + b_1\sin x)e^{-\kappa t} + (a_2\cos 2x + b_2\sin 2x)e^{-\kappa 2^2 t} + \ldots$$
or
$$\sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx)e^{-\kappa n^2 t}.$$

It is clear that each term of this series satisfies the differential equation (1) and the conditions (3), and that if we were dealing with the sum of a finite number of terms, the sum would also satisfy these conditions. In the case of an infinite series we have seen * that we must proceed with more caution.

Since $f(x)$ is bounded,† there is a positive number M such that $|f(x)| < M$ in $(-\pi, \pi)$. It follows that $|a_0| < M$, $|a_n| < 2M$ and $|b_n| < 2M$ for all values of n.

Therefore $\quad |(a_n\cos nx + b_n\sin nx)e^{-\kappa n^2 t}| < 4Me^{-\kappa n^2 t_0},$

where
$$t \geqq t_0 < 0.$$

But the series
$$\sum_{n=0}^{\infty} e^{-\kappa n^2 t_0}$$
is convergent, and its terms are independent both of x and t; therefore the series
$$\sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx)e^{-\kappa n^2 t}$$
is uniformly convergent for any interval of x, when $t > 0$, and regarded as a function of t, it is uniformly convergent when $t \geqq t_0 > 0$, t_0 being any positive number.

The function v defined by this series is thus a continuous function of x and a continuous function of t in these intervals. (Cf. *F.S.*, § 68.)

It is easy to show that the series we obtain by term by term differentiation of v with respect to x and t are also uniformly convergent in these intervals of x and t respectively. Thus these series represent the differential coefficients of the function v. (Cf. *F.S.*, § 71.)

* Cf. *F.S.*, Ch. V. † Cf. *F.S.*, §§ 24, 31.

Hence $$\frac{\partial v}{\partial t}=-\sum_{n=0}^{\infty}\kappa n^2(a_n\cos nx+b_n\sin nx)e^{-\kappa n^2 t},$$

$$\kappa\frac{\partial^2 v}{\partial x^2}=-\sum_{n=0}^{\infty}\kappa n^2(a_n\cos nx+b_n\sin nx)e^{-\kappa n^2 t},$$

and $$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}.$$

Thus v satisfies the differential equation (1).

Now the series which defines v has been shown to be uniformly convergent for any interval of x, when $t>0$, and thus to be continuous in such an interval, and the same holds of the series for $\frac{\partial v}{\partial x}$.

But the values we obtain for v when we substitute $x=\pm\pi$ are the same. Therefore $$\operatorname*{Lt}_{x\to\pi}(v)=\operatorname*{Lt}_{x\to-\pi}(v),$$

and similarly $$\operatorname*{Lt}_{x\to\pi}\left(\frac{\partial v}{\partial x}\right)=\operatorname*{Lt}_{x\to-\pi}\left(\frac{\partial v}{\partial x}\right).$$

It would have been more correct to state the conditions (3) of our problem in this form, since we are not so much concerned with the value of these functions for $x=\pm\pi$ as with their limits when x tends to $\pm\pi$.

We have now to examine whether the function v satisfies the initial conditions (2). For this purpose we must use the extension of Abel's Theorem (cf. *F.S.*, § 73, I.), since we have only proved that the series for v is uniformly convergent when $t\geqq t_0>0$, and without further examination we could not use the fact that $v=f(x)$ when $t=0$ as equivalent to the initial condition (2), which is really that $$\operatorname*{Lt}_{t\to 0}(v)=f(x).$$

In the extension of Abel's Theorem above referred to, we saw that when $$a_0+a_1+a_2+\dots$$
is a convergent series whose sum is A, then the series
$$\phi(t)=a_0e^{-a_0t}+a_1e^{-a_1t}+\dots,$$
where $0\leqq a_0<a_1<\dots$ and $0<t$, is also a convergent series, and
$$\operatorname*{Lt}_{t\to 0}\phi(t)=a_0+a_1+\dots=A.$$

Let us apply this theorem to the series
$$v=a_0+(a_1\cos x+b_1\sin x)e^{-\kappa t}+\dots.$$

Since we have assumed that the continuous function $f(x)$ satisfies Dirichlet's Conditions in the interval $-\pi \leqq x \leqq \pi$, and that $f(\pi)=f(-\pi)$, we know that the series

$$a_0+(a_1 \cos x+b_1 \sin x)+\ldots$$

converges to the value $f(x)$ in the whole interval $-\pi \leqq x \leqq \pi$. (Cf. *F.S.*, § 95.)

Thus our series converges when $t=0$, and our theorem tells us that

$$\underset{t \to 0}{\mathrm{Lt}}(v)=f(x).$$

Therefore $$v=\sum_{n=0}^{\infty}(a_n \cos nx+b_n \sin nx)e^{-\kappa n^2 t}$$

satisfies all the conditions of the problem.

13. The Steady Temperature of the Ring.

Let the section of the ring at $x=\pm\pi$ be maintained at a constant temperature V until the flow of heat has become stationary along the ring, radiation taking place into a medium at a constant temperature, which we take as the zero of our scale.

The equations for v are

$$(1)\quad \frac{d^2v}{dx^2}-\mu^2 v=0, \text{ where } \mu^2=\frac{Hp}{K\omega}, \ (-\pi<x<\pi)$$

$$(2)\quad v=V \text{ at } x=\pm\pi,$$

and $$(3)\quad \frac{dv}{dx}=0 \text{ at } x=0,$$

the last equation being required by the symmetry of the distribution of temperature.

The general solution of (1) is

$$v=A\cosh(\mu x+a),$$

A and a being arbitrary constants, and it is clear that all the conditions are satisfied by the solution

$$v=V\frac{\cosh \mu x}{\cosh \mu\pi},$$

which thus expresses the final state of temperature in the ring.

A method of determining the conductivity is founded upon this result.*

* Cf. § 32.

Let the temperatures be v_1, v_2 and v_3 at any three points x_1, x_2 and x_3 in the ring, and let

$$x_2-x_1=x_3-x_2=l.$$

Then
$$\frac{v_1+v_3}{v_2}=\frac{\cosh\mu x_1+\cosh\mu x_3}{\cosh\mu x_2}=2\cosh\mu l,$$

and for three such points this ratio is constant. This result is confirmed by experiment, and was first pointed out by Fourier.*

Putting $(v_1+v_3)/v_2=2n$, say, we have $e^{\mu l}=n+\sqrt{(n^2-1)}$,

and thus
$$l\sqrt{\frac{Hp}{K\omega}}=\log(n+\sqrt{(n^2-1)}).$$

If, then, we have two rings of equal perimeter, cross-section and emissivity, and temperature observations are taken at three points as described above, we would obtain the ratio of their conductivities in the form
$$\frac{K_2}{K_1}=\left(\frac{\log(n_1+\sqrt{(n_1^2-1)})}{\log(n_2+\sqrt{(n_2^2-1)})}\right)^2,$$

$2n_1$ and $2n_2$ being the values of $(v_1+v_3)/v_2$ in the two substances.

The disadvantage of this method is the uncertain character of the emissivity.

14. Neumann's Ring Method of obtaining the Values of the Conductivity and Emissivity.

Suppose the ring, as in § 13, heated at $x=\pm\pi$ until the flow of heat has become steady. The source of heat is then removed, and the ring is allowed to cool, radiation taking place into a medium at constant temperature, which we shall take as zero. Measuring the time from the instant at which the source of heat is removed, the equations for v are as follows:

$$(1)\qquad \frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-\lambda v,\quad (t>0,\ -\pi<x<\pi)$$

$$(2)\qquad v=V\frac{\cosh\mu x}{\cosh\mu\pi},\quad (t=0,\ -\pi\leqq x\leqq\pi)$$

$$(3)\qquad \left.\begin{aligned} v_{x=\pi}&=v_{x=-\pi}\\ \left(\frac{\partial v}{\partial x}\right)_{x=\pi}&=\left(\frac{\partial v}{\partial x}\right)_{x=-\pi}\end{aligned}\right\},\quad (t>0)$$

where
$$\lambda=\frac{Hp}{c\rho\omega},\quad \kappa=\frac{K}{c\rho}\quad\text{and}\quad \mu=\sqrt{\frac{\lambda}{\kappa}}.$$

* Fourier, *loc. cit.*, §§ 107-110.

Putting $v=e^{-\lambda t}u$, these equations give

$$(4)\qquad \frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2}, \qquad (t>0,\ -\pi<x<\pi)$$

$$(5)\qquad u=V\frac{\cosh \mu x}{\cosh \mu\pi}, \quad (t=0,\ -\pi\leqq x\leqq\pi)$$

$$(6)\qquad \left.\begin{aligned} u_{x=\pi}&=u_{x=-\pi}\\ \left(\frac{\partial u}{\partial x}\right)_{x=\pi}&=\left(\frac{\partial u}{\partial x}\right)_{x=-\pi}\end{aligned}\right\}. \quad (t>0)$$

But, using the Cosine Series for $\dfrac{\cosh \mu x}{\cosh \mu\pi}$, namely,

$$\frac{\cosh \mu x}{\cosh \mu\pi}=\frac{2\mu \tanh \mu\pi}{\pi}\left[\frac{1}{2\mu^2}+\sum_1^\infty\frac{\cos n\pi}{\mu^2+n^2}\cos nx\right],$$

our solution for v follows at once, and is given by

$$v=\frac{2V\mu}{\pi}\tanh \mu\pi e^{-\lambda t}\left[\frac{1}{2\mu^2}+\sum_1^\infty\frac{\cos n\pi}{\mu^2+n^2}\cos nx e^{-\kappa n^2 t}\right].$$

After a considerable time has passed, the convergency of this series becomes very rapid owing to the presence of the factor $e^{-\lambda t-\kappa n^2 t}$.

Neglecting the terms after the first two, we have the equations,

$$v_0+v_\pi=\frac{2V}{\mu\pi}\tanh \mu\pi\, e^{-\lambda t},$$

$$-v_0+v_\pi=\frac{4V\mu}{(\mu^2+1)\pi}\tanh \mu\pi\, e^{-(\kappa+\lambda)t},$$

connecting the temperatures at $x=0$ and $x=\pi$.

This method requires the observations of the temperature when the ring is cooling at the points $x=0$ and $x=\pi$. These observations should be taken at equidistant intervals after a sufficient time has passed to allow our approximations to hold. If these conditions are satisfied, the observed values of $\log(v_\pi\pm v_0)$ will lie on two straight lines.

Let
$$v_0+v_\pi=a_1 \text{ when } t=t_1,$$
$$v_0+v_\pi=a_2 \text{ when } t=t_2.$$

Then
$$\frac{a_1}{a_2}=e^{\lambda(t_2-t_1)},$$

and
$$\lambda=\frac{\log a_2-\log a_1}{t_1-t_2}.$$

The mean of a set of such observations will thus give the value of λ.

In the same way, let

$$-v_0+v_r=b_1 \text{ when } t=t_1,$$
$$-v_0+v_r=b_2 \text{ when } t=t_2.$$

Then
$$\frac{b_1}{b_2}=e^{(\kappa+\lambda)(t_2-t_1)},$$

and
$$\kappa+\lambda=\frac{\log b_2-\log b_1}{t_1-t_2}.$$

When κ and λ have been determined, the values of K and H follow.*

* Neumann, *Ann. chim. phys., Paris* (Sér. 3), 66, p. 183, 1862; *Phil. Mag., London* (Ser. 4), 25, p. 63, 1863; Kirchhoff, *Vorlesungen über mathematische Physik*, Bd. IV., p. 40, Leipzig, 1894.

CHAPTER III

THE LINEAR FLOW OF HEAT. THE INFINITE AND SEMI-INFINITE SOLID AND ROD

15. Introductory.

In this chapter we shall examine the different problems where the isothermal surfaces are planes parallel to $x=0$ and the flow of heat is linear, the lines of flow being parallel to the axis of x. It will be seen that the results we obtain in this way also serve for the flow of heat along straight rods of small cross-section when there is no radiation at the surface.

After obtaining the solution for the Infinite Solid, we proceed to examine, in detail, the many important problems of Linear Flow of Heat in the Semi-Infinite Solid, or the solid which is bounded by the plane $x=0$ and extends to infinity in the direction of x positive. Various applications of these results in obtaining the values of the Conductivity will be noticed. The corresponding problems in the case of the Finite Solid bounded by the planes $x=0$ and $x=a$ will be treated in the next chapter.

16. The Infinite Solid.

In the theoretical case where the solid is unbounded and the initial temperature is given by the equation

$$v=f(x),$$

the equation of conduction reduces to

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},$$

since v depends only on x and t.

Consider the expression

$$u=\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4\kappa t}}.$$

Since $$\frac{\partial u}{\partial t} = -\frac{1}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4\kappa t}} + \frac{x^2}{4\kappa t^{\frac{5}{2}}} e^{-\frac{x^2}{4\kappa t}}$$

and $$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2\kappa t^{\frac{3}{2}}} e^{-\frac{x^2}{4\kappa t}} + \frac{x^2}{4\kappa^2 t^{\frac{5}{2}}} e^{-\frac{x^2}{4\kappa t}},$$

this expression is a particular integral of the differential equation,

Therefore $$\frac{1}{2\sqrt{(\pi\kappa t)}} e^{-\frac{(x-x')^2}{4\kappa t}}$$

is also an integral.

Further, the equation being linear, the sum of any number of particular integrals is also an integral, and thus

$$v = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{-\infty}^{\infty} f(x') e^{-\frac{(x-x')^2}{4\kappa t}} dx'$$

satisfies the equation, assuming that this integral is convergent.

Putting $$x' = x + 2\sqrt{(\kappa t)}\,\xi,$$

we find that $$v = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\sqrt{(\kappa t)}\,\xi)\, e^{-\xi^2} d\xi.$$

In the limit when $t \to 0$, $f(x+2\sqrt{(\kappa t)}\xi) = f(x)$, if this function is continuous; and it is assumed that the limiting value of this integral is given by

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-\xi^2} d\xi,$$

which is equal to $f(x)$.

Therefore the temperature in the Infinite Solid at time t, due to the initial temperature

$$v = f(x),$$

is given by $$v = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{-\infty}^{\infty} f(x') e^{-\frac{(x-x')^2}{4\kappa t}} dx'.$$

The corresponding results for two and three dimensions are

$$v = \frac{1}{4\pi\kappa t} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y') e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}} dx'\, dy'$$

and $$v = \frac{1}{(2\sqrt{(\pi\kappa t)})^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y', z') e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}} dx'\, dy'\, dz'.$$

Since $$\int_0^{\infty} e^{-a^2x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{a^2}},\text{*}$$

* Cf. *F.S.*, p. 195, Ex. 13, and Gibson, *Treatise on the Calculus* (2nd. Ed.), p. 46[illegible]

and therefore $\int_0^\infty e^{-\kappa a^2 t}\cos a(x'-x)\,da = \frac{\sqrt{\pi}}{2\sqrt{(\kappa t)}}e^{-\frac{(x'-x)^2}{4\kappa t}}$,

we may transform the expression for v into

$$\frac{1}{\pi}\int_{-\infty}^{\infty}dx'\int_0^\infty f(x')\cos a(x'-x)\,e^{-\kappa a^2 t}\,da,$$

a form which would be suggested by Fourier's Integral for $f(x)$, namely,

$$\frac{1}{\pi}\int_0^\infty da\int_{-\infty}^{\infty}f(x')\cos a(x'-x)\,dx'.$$

17. The above is the form in which Laplace's solution for the Infinite Solid is usually presented. There are several points in the argument which obviously require fuller treatment if the discussion is to be at all rigorous.*

I. We shall assume, in the first place, that the arbitrary function $f(x)$ is bounded for all values of x {*e.g.* $|f(x)|<M$, for all values of x} and integrable in any given interval.

Let $$v(x, t)=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{-\infty}^{\infty}f(x')e^{-\frac{(x-x')^2}{4\kappa t}}\,dx'.$$

This integral is convergent when $t>0$, and it can be differentiated under the sign of integration, both with regard to x and t. (Cf. *F.S.*, § 86).

Let x be a point at which the arbitrary function is continuous.

Then to the positive number ϵ, chosen as small as we please, there corresponds a positive number η such that

$$|f(x')-f(x)|<\tfrac{1}{4}\epsilon, \text{ when } |x'-x|\leqq\eta.$$

Also $$v(x, t)=\frac{1}{2\sqrt{(\pi\kappa t)}}\left(\int_{-\infty}^{x-\eta}+\int_{x-\eta}^{x+\eta}+\int_{x+\eta}^{\infty}\right)f(x')\,e^{-\frac{(x'-x)^2}{4\kappa t}}\,dx'.$$

Denote these integrals by I_1, I_2 and I_3.

Then $$I_1=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{-\infty}^{x-\eta}f(x')\,e^{-\frac{(x'-x)^2}{4\kappa t}}\,dx'$$
$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\eta/2\sqrt{(\kappa t)}}f(x+2\sqrt{(\kappa t)}\,u)\,e^{-u^2}\,du.$$

Therefore $$|I_1|<\frac{M}{\sqrt{\pi}}\int_{-\infty}^{-\eta/2\sqrt{(\kappa t)}}e^{-u^2}\,du.$$

Now η is known: it follows that we can choose t_1 so that

$$\int_{-\infty}^{-\eta/2\sqrt{(\kappa t)}}e^{-u^2}\,du<\frac{\sqrt{\pi}}{4M}\epsilon, \text{ when } 0<t\leqq t_1,$$

since the integral $$\int_{-\infty}^{x}e^{-u^2}\,du \text{ converges.}$$

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* For the bearing of this work on the representation of continuous functions by a series of polynomials, reference may be made to Borel, *Leçons sur les fonctions de variables réelles*, p. 50, Paris, 1905.

Thus $$|I_1|<\tfrac{1}{3}\epsilon, \text{ when } 0<t\leqq t_1.$$

Similarly, we can choose t_3 so that
$$|I_3|<\tfrac{1}{3}\epsilon, \text{ when } 0<t\leqq t_3.$$

Further,

$$I_2=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}f(x')e^{-\frac{(x'-x)^2}{4\kappa t}}dx'$$
$$=\frac{f(x)}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}e^{-\frac{(x'-x)^2}{4\kappa t}}dx'+\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}(f(x')-f(x))e^{-\frac{(x'-x)^2}{4\kappa t}}dx'$$
$$=\frac{f(x)}{\sqrt{\pi}}\int_{-\eta/2\sqrt{(\kappa t)}}^{\eta/2\sqrt{(\kappa t)}}e^{-u^2}du+\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}(f(x')-f(x))e^{-\frac{(x'-x)^2}{4\kappa t}}dx'$$
$$=\frac{2f(x)}{\sqrt{\pi}}\left(\int_0^\infty e^{-u^2}du-\int_{\eta/2\sqrt{(\kappa t)}}^\infty e^{-u^2}du\right)$$
$$+\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}(f(x')-f(x))e^{-\frac{(x'-x)^2}{4\kappa t}}dx'.$$

Therefore

$$|I_2-f(x)|<\frac{2|f(x)|}{\sqrt{\pi}}\int_{\eta/2\sqrt{(\kappa t)}}^\infty e^{-u^2}du+\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}|f(x')-f(x)|e^{-\frac{(x'-x)^2}{4\kappa t}}$$

But we can choose t_2 so that

$$\frac{2|f(x)|}{\sqrt{\pi}}\int_{\eta/2\sqrt{(\kappa t)}}^\infty e^{-u^2}du<\tfrac{1}{6}\epsilon, \text{ when } 0<t\leqq t_2.$$

Also $$\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{x-\eta}^{x+\eta}|f(x')-f(x)|e^{-\frac{(x'-x)^2}{4\kappa t}}dx'<\frac{\epsilon}{4\sqrt{\pi}}\int_{-\eta/2\sqrt{(\kappa t)}}^{\eta/2\sqrt{(\kappa t)}}e^{-u^2}du$$
$$<\tfrac{1}{6}\epsilon, \text{ when } t>0.$$

Therefore $$|I_2-f(x)|<\tfrac{1}{3}\epsilon, \text{ when } 0<t\leqq t_2.$$

But $$v(x,t)-f(x)=I_1+(I_2-f(x))+I_3.$$

Thus, if τ is the smallest of t_1, t_2 and t_3, we have
$$|v(x,t)-f(x)|<\epsilon, \text{ when } 0<t\leqq\tau.$$

In other words, *we have shown that*
$$\underset{t\to 0}{\mathrm{Lt}}\ v(x,t)=f(x),$$
when x is any point at which $f(x)$ is continuous, and the function has been assu bounded for all values of x and integrable in any given interval.

II. It will be found by a similar argument that
$$\underset{t\to 0}{\mathrm{Lt}}\ v(x,t)=\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$
when the limits $f(x+0)$ and $f(x-0)$ exist, and the function is subjec the same conditions as before.

III. Let $f(x)$ be continuous in the interval (a,β) and also at the ends of interval. Then the number η referred to above will serve for all value x such that $a\leqq x\leqq\beta$. (Cf. *F.S.*, § 31, Theorem I.)

With some obvious verbal changes in (I.) it will be seen that $v(x,t)$ te uniformly to $f(x)$ in the interval (a,β) as $t\to 0$.

In other words, we have

$$|v(x, t)-f(x)|<\epsilon, \text{ when } 0<t\leqq\tau,$$

the same τ serving for all values of x in (α, β).

IV. The theorem established in (I.) is also true, when the arbitrary function $f(x)$ does not satisfy all the conditions there imposed upon it.

For example, if $f(x)=x^2$, it is not bounded for all values of x.

But
$$\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{-\infty}^{\infty} x'^2 e^{-\frac{(x'-x)^2}{4\kappa t}}\, dx' = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(x+2\sqrt{(\kappa t)}u)^2 e^{-u^2}du$$
$$=x^2+\frac{4\kappa t}{\sqrt{\pi}}\int_{-\infty}^{\infty}u^2e^{-u^2}du.$$

It follows that when $t\to 0$, $\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{-\infty}^{\infty} x'^2 e^{-\frac{(x'-x)^2}{4\kappa t}}dx'$ has the limit x^2.

Further, it will be seen that when

$$f(x)=a_0+a_1x+\ldots+a_nx^n,$$

and
$$v(x, t)=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{-\infty}^{\infty} f(x')e^{-\frac{(x'-x)^2}{4\kappa t}}\, dx',$$

we also have
$$\operatorname*{Lt}_{t\to 0} v(x, t)=f(x).^*$$

V. It may be noted that in the above discussion it has not been assumed that $\int_{-\infty}^{\infty} f(x')dx'$ converges. It is not difficult to show, as in (I.), that *when $f(x)$ is bounded and integrable in any given interval, and $\int_{-\infty}^{\infty}|f(x')|dx'$ converges, $v(x, t)$ has the limit $f(x)$ (or $\frac{1}{2}[f(x+0)+f(x-0)]$) as $t\to 0$, when the function is continuous at x or has an ordinary discontinuity there.*

18. The Semi-Infinite Solid.

Let the solid be bounded by the plane $x=0$ and extend to infinity in the direction of x positive, the initial temperature being given by $v=f(x)$, and the plane $x=0$ being kept at zero temperature. The solution of this problem may be deduced from that of the Infinite Solid.

We suppose the solid continued on the negative side of the plane $x=0$, and the initial temperature at $-x'$ $(x'>0)$ to be $-f(x')$, the initial temperature at x' being $f(x')$. With this distribution the plane $x=0$ will remain at zero.

Then we have

$$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\left(\int_0^{\infty} f(x')e^{-\frac{(x-x')^2}{4\kappa t}}\, dx'+\int_{-\infty}^{0}(-f(-x'))e^{-\frac{(x-x')^2}{4\kappa t}}\, dx'\right),$$

and this reduces to

$$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^{\infty} f(x')\left(e^{-\frac{(x-x')^2}{4\kappa t}}-e^{-\frac{(x+x')^2}{4\kappa t}}\right)dx'.$$

* For a more general discussion see Goursat, *Cours d'Analyse*, T. III., § 543, 1915.

It is clear that this value of v satisfies all the conditions of problem of the Semi-Infinite Solid whose bounding plane is k at zero temperature.

When the initial temperature is a constant, V, this express may be simplified by substituting $x'=x+2\sqrt{(\kappa t)}\xi$ in the first p and $x'=-x+2\sqrt{(\kappa t)}\xi$ in the second.

We thus obtain
$$v=\frac{V}{\sqrt{\pi}}\int_{-x/2\sqrt{(\kappa t)}}^{x/2\sqrt{(\kappa t)}} e^{-\xi^2}\,d\xi$$
$$=\frac{2V}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}} e^{-\xi^2}\,d\xi.$$

The definite integrals of this type have been tabulated,* an we write
$$\Theta(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,d\xi,$$
the solution of the problem of the Semi-Infinite Solid, whose sur is kept at zero temperature, the initial temperature being given by
$$v=V\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right).$$

With the aid of the tables for these functions we can find time which must elapse before the temperature at a depth x fallen to a given fraction—say $\frac{1}{2}$—of its original value.

Since, if $v=\frac{1}{2}V$, we must have
$$\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right)=\cdot 5,$$
and from the tables for $\Theta(x)$ it follows that
$$\frac{x}{2\sqrt{(\kappa t)}}=\cdot 477 \text{ approximately.}$$

From calculations based upon the values of κ for silver bismuth, Weber† states that it would take $\frac{1}{4}$ second for the t peratures to fall by one half at a depth of 1 cm. in silver, and

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* The first table of these integrals was published by Encke in a paper o "Method of Least Squares" in the Berlin *Astronomisches Jahrbuch* for giving the values of $\Theta(x)$ for $x=0$ to $x=2$ at intervals of ·01 computed to decimal places. De Morgan extended this to $x=3$ in his "Essay on Probabilit (1838). A new table, to fifteen places, from $x=0$ to $x=3$ at intervals of ·000 been published by Burgess in his paper "On the Definite Integrals $\frac{2}{\sqrt{\pi}}\int_0^t$ with extended Tables of Values," *Edinburgh, Trans. R. Soc.*, 39, p. 257, 189

† Weber-Riemann, *Die partiellen Differential-gleichungen der mathemati Physik*, Bd. II. (2 Aufl.), § 37, Braunschweig, 1912.

in bismuth it would take 8 minutes: while the times required for such a change at a depth of 1 metre would be about 1 hour and $1\frac{1}{4}$ months respectively in the two substances.

It will be noticed that the expression for the temperature may be transformed as in § 16 into

$$v=\frac{1}{\pi}\int_0^\infty dx'\int_0^\infty f(x')\left[\cos a(x'-x)-\cos a(x'+x)\right]e^{-\kappa a^2 t}\,da$$

$$=\frac{2}{\pi}\int_0^\infty dx'\int_0^\infty f(x')\sin ax'\sin ax\,e^{-\kappa a^2 t}\,da,$$

a form suggested by Fourier's Integral

$$f(x)=\frac{2}{\pi}\int_0^\infty da\int_0^\infty f(x')\sin ax'\sin ax\,dx'.$$

Ex. 1. Prove that when the boundary $x=0$ is kept at temperature unity and the initial temperature is zero,

$$v=1-\frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-\xi^2}\,d\xi.$$

Ex. 2. Prove that when the boundary $x=0$ is impervious to heat, the solution takes the form

$$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^\infty f(x')\left(e^{-\frac{(x-x')^2}{4\kappa t}}+e^{-\frac{(x+x')^2}{4\kappa t}}\right)dx'$$

$$=\frac{2}{\pi}\int_0^\infty dx'\int_0^\infty f(x')\cos ax'\cos ax\,e^{-\kappa a^2 t}\,da.$$

The curves

$$v=\frac{2V}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-\xi^2}\,d\xi$$

for different values of t may be drawn. As t increases these curves get flatter and approach the line $v=0$. As t gets smaller and smaller they approach the line

$$v=V,\ \text{when}\ x>0,$$

and the limiting form of the curve for $t=0$ is the origin

$$\left.\begin{matrix}v=0\\x=0\end{matrix}\right\}\ \text{and the line}\ \left.\begin{matrix}v=V\\x>0\end{matrix}\right\}.$$

If we take the case in which the semi-infinite solid is initially at zero temperature and the surface $x=0$ is kept at unit temperature, the solution is given by

$$v=1-\frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-\xi^2}\,d\xi,$$

and in this case

$$\frac{\partial v}{\partial t}=\frac{x}{2\sqrt{(\pi\kappa t^3)}}\,e^{-\frac{x^2}{4\kappa t}}.$$

We are indebted to Professor A. Stanley Mackenzie for permission to reproduce the curves, Figs. 4, 5, 9, and 10, given in his paper "On Some Equations pertaining to the Propagation of Heat in an Infinite Medium" *Philadelphia, Pa., Proc. Amer. Phil. Soc.*, 41, 1902).

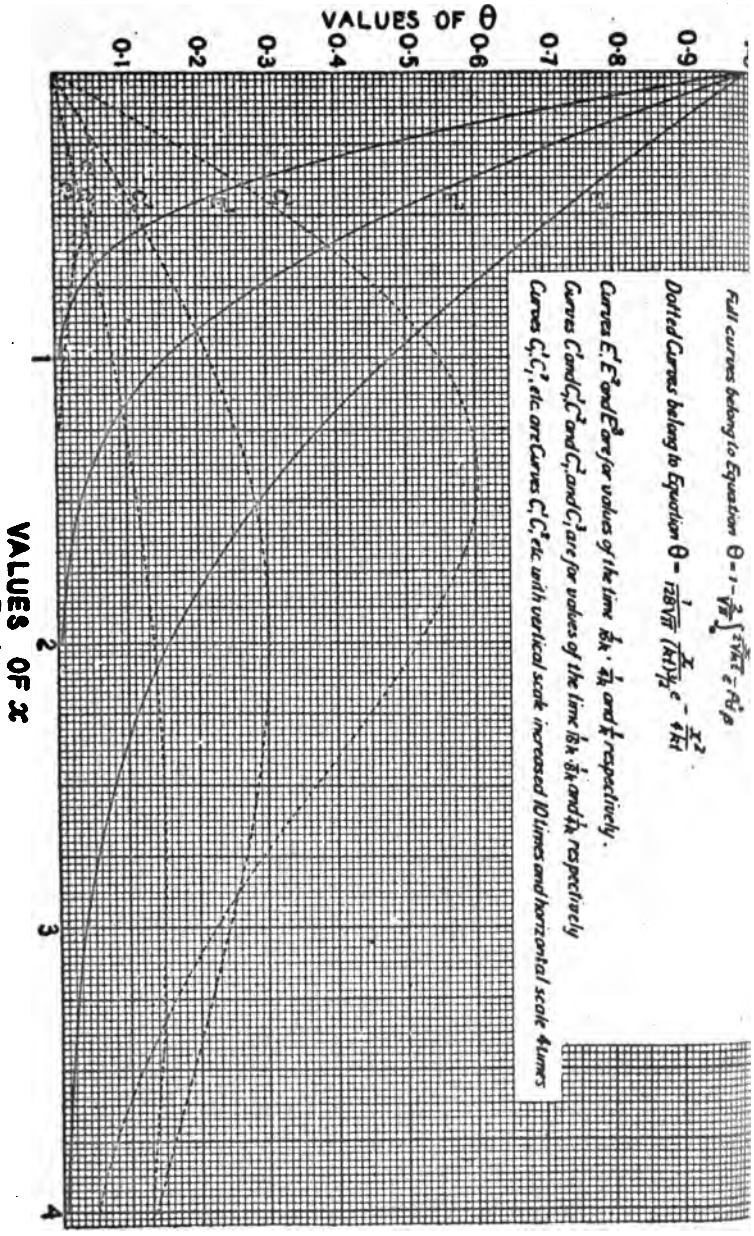

FIG. 4.

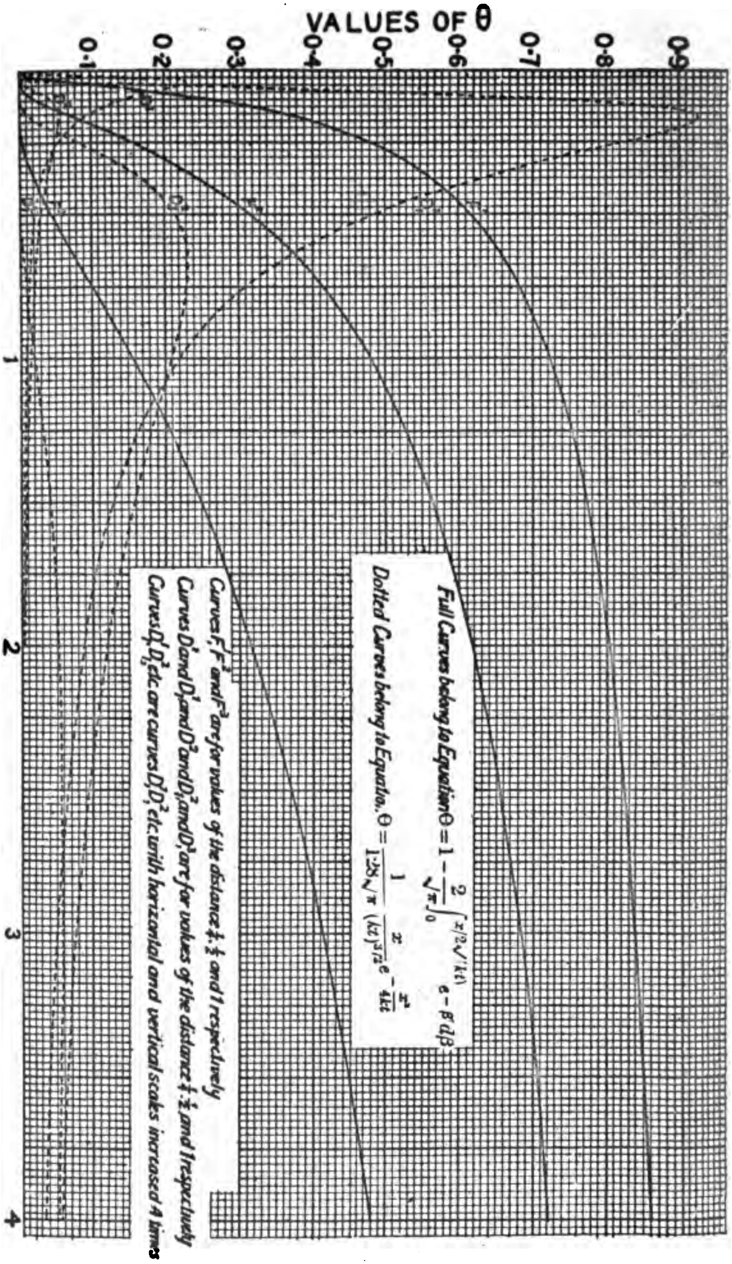
VALUES OF θ
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
Full Curves belong to Equation $\theta = 1 - \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{kt}} e^{-\beta^2}\,d\beta$
Dotted Curves belong to Equation. $\theta = \frac{1}{1{\cdot}28\sqrt{\pi}}\frac{x}{(kt)^{3/2}}e^{-\frac{x^2}{4kt}}$
Curves F, F² and F³ are for values of the distance ¼, ½ and 1 respectively
Curves D and D₁, D² and D₁², and D³ are for values of the distance ¼, ½ and 1 respectively
Curves D₅, D₅¹, D₅² etc. are curves D, D¹, D² etc. with horizontal and vertical scales increased 4 times
1
2
3
4
VALUES OF 4 kt

FIG. 5.

19. The Infinite or Semi-Infinite Rod.

The problems for the Infinite or Semi-Infinite Rod of small cross-section may be solved in the same way. As in the case of Fourier's Ring, the cross-section of the rod is supposed so small that the temperature at all points of the section may be considered the same as that at its centre.

Suppose the rod to lie along the axis of x, and consider the element of volume bounded by the sections at $P(x)$ and $P'(x+dx)$.

The rate at which heat flows into this element over the face at P is

$$-K\frac{\partial v}{\partial x}\omega,$$

where ω is the area of the cross-section of the rod.

Similarly, the rate at which heat flows across the face at P' is

$$\left(-K\frac{\partial v}{\partial x}-K\frac{\partial^2 v}{\partial x^2}dx-\dots\right)\omega.$$

Hence ultimately the rate of gain of heat in the element from these two faces is

$$\omega K\frac{\partial^2 v}{\partial x^2}dx.$$

The rate at which heat is lost by radiation at the surface is

$$H(v-v_0)p\,dx,$$

where p is the perimeter of the cross-section and v_0 is the temperature of the medium.

Also the total rate of gain of heat in the element is ultimately

$$\omega c\rho\frac{\partial v}{\partial t}dx.$$

Thus we have $$\frac{\partial v}{\partial t}=\frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2}-\frac{Hp}{c\rho\omega}(v-v_0),$$

which becomes $$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-\lambda(v-v_0)$$

on putting $$\frac{Hp}{c\rho\omega}=\lambda \quad\text{and}\quad \frac{K}{c\rho}=\kappa.$$

When the surface of the rod is rendered impervious to heat, so that no radiation takes place, the equation for the temperature takes the form

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},$$

and the problems on the distribution of temperature in an Infinite

or Semi-Infinite Rod are reduced to those of Linear Flow in an Infinite or Semi-Infinite Solid.

When radiation takes place into a medium at constant temperature, this may be taken as the zero of our scale, and the equation becomes

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-\lambda v,$$

which reduces to

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2},$$

on substituting $v=ue^{-\lambda t}$.

Thus the problem is reduced to that of Linear Flow already examined.

If the material of the rod is not homogeneous, it is clear that the temperature equation becomes

$$\frac{\partial v}{\partial t}=\frac{1}{c\rho}\frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x}\right)-\frac{Hp}{c\rho\omega}(v-v_0).$$

20. Conductivity Experiments upon Bars. Steady Temperature.

The fundamental experiment described in § 2, from which our definition of conductivity is derived, has been used in the determination of the conductivities of different substances. The mathematical theory of the Conduction of Heat in a Semi-Infinite Rod has also been employed in finding the conductivity and emissivity. We shall refer in this article to experiments in which the Steady Temperature is used.

A straight bar of small cross-section of the material to be tested is taken and heated at one end till the temperature becomes steady. If the bar is long enough, the temperature of the further end is practically unaffected by the source of heat and remains the same as that of the surrounding medium, which is taken as zero. The circumstances of the experiment are thus represented by the equations

$$\kappa\frac{d^2 v}{dx^2}-\lambda v=0,\quad v=V \text{ at } x=0,\quad v=0 \text{ at } x=\infty,$$

and

$$v=Ve^{-\sqrt{(\lambda/\kappa)}x}$$

gives the temperature at the distance x from the end which is kept at the temperature V long enough for the steady state of temperature to be reached. It will be noticed that this requires that the rod should be of such a length l that $\sqrt{(\lambda/\kappa)}\,l$ is very large.

Bars of different metals of the same dimensions are used and the surfaces are varnished in the same way, so that the value of the emissivity will be the same for each. In this case temperature observations in the rods give the values of λ/κ, and thus the ratios of the conductivities are obtained. The experiments of Ingenhausen, Despretz, Wiedemann and Franz, are based upon this method, and descriptions of their work may be found in Text-books of Physics.*

However the power to radiate heat is one which it is hard to regulate, and for this reason these experiments are not of such value as others which we shall discuss later, in which the conductivity is found directly and without reference to the value of the emissivity. It is to be noticed also that these experiments only give the relative values of the conductivities.

The classical experiments of Forbes† afford a method of obtaining the absolute value of the conductivity of metals. Forbes also employed the Bar Method and used the Steady State of Temperature. His method consists of two essentially distinct sets of observations. In the first, the steady state of temperature of a long bar of wrought iron (8 ft. long and $1\frac{1}{4}$ sq. in. in cross-section) was considered. The bar was heated at one end till the temperature had become steady, and it was of sufficient length to allow the end furthest from the source of heat to keep the temperature of the surrounding medium. The rate of flow of heat across the section distant x from the heated end is given by

$$-K\omega\frac{\partial v}{\partial x},$$

ω being the area of the cross-section.

This must be the same as the rate at which heat is being lost by radiation at the surface of the bar from this section to the end.

Forbes determined the value of $\dfrac{\partial v}{\partial x}$ from the readings of thermometers placed at different points along the bar, and his work is thus independent of the mathematical solution which the other experimenters employed.

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* Cf. Poynting and Thomson, *Text-book of Physics—Heat* (6th Ed.), p. 96 *et seq.* Preston, *Theory of Heat* (3rd Ed.), §§ 296-299; Winkelmann, *Handbuch der Physi*[k] (2. Aufl.), Bd. III., p. 450 *et seq.*

† *Edinburgh, Trans. R. Soc.*, 23, p. 133, 1864.

The second set of observations was designed to give the rate at which heat was being lost at the surface.

For this purpose he employed another bar of the same material, and exactly similar to the first, except that its length was only 20 inches. This bar he heated uniformly and placed to cool alongside the other, which had now cooled, so that the circumstances of the radiation might be the same. The rate at which heat was radiated by this bar at different temperatures could be obtained, since for any element dx it would be equal to

$$-\omega c\rho\, dx \frac{\partial v}{\partial t}.$$

But the value of $\frac{\partial v}{\partial t}$ corresponding to any definite temperature is given by continued observations of the temperature at any point. Thus the quantity of heat lost per second by every part of the larger bar in the first part of the experiment could be determined.

Therefore K is given by the equation

$$K \frac{\partial v}{\partial x} = c\rho \int_x^l \frac{\partial v}{\partial t} dx.$$

In this work Forbes employed graphical methods, $\frac{\partial v}{\partial x}$ on the left-hand side of this equation being obtained from the curve of the temperature given by the observations on the long bar, and the integral on the right-hand side being obtained as the area of a curve plotted from the temperature observations on the second bar.

By these means he found the value of the conductivity at different sections of the bar at different temperatures, and he showed that the conductivity of iron decreases with rise of temperature. His observations have been repeated by different physicists, and they occupy an important position among the methods for determining the value of the conductivity of metals.*

21. Conductivity Experiments upon Bars (*continued*).

Variable Temperature. Ångström's Method.†

In the preceding article we have shown how the Steady Temperature of a long metal rod of small cross-section may be employed

* Cf. Poynting and Thomson, *loc. cit.*, p. 98; Winkelmann, *loc. cit.*, p. 454.

† *Ann. Physik, Leipzig*, 114, p. 513, 1861; 123, p. 628, 1864.

in obtaining the conductivity of the substance. The variabl temperature has also been used, in the case in which one end of th bar is subjected to periodic variations of temperature, which cau heat waves to travel down the bar. The conductivity is calculat from the march of these waves. Ångström was the first to emplo this method, and his work is of exceptional interest both from th neatness of the mathematical discussion and the novelty of h experimental method. Hagström * later discussed the same pr blem, assuming that the conductivity and emissivity vary with th temperature; Neumann and Weber † extended the method to th case of a short bar, both ends of which undergo periodic chang of temperature.

Ångström employed long bars of small cross-section. The e $x=0$ was subjected to periodic changes of temperature, bei alternately heated by a current of steam and cooled by a curre of cold water for equal intervals. When this has gone on for so time, the temperature in the bar will ultimately settle down to periodic state, independent of the initial distribution. It is th periodic state which Ångström investigates and upon which h results depend. The bar is allowed to radiate into a medium a constant temperature, taken as the zero of the experiment. before, it is supposed of such small cross-section that the temperat over the section may be taken as that at the centre, and of su length that the temperature at the further end remains unaffect by the alterations at $x=0$, so that in the mathematical treatm it is supposed unlimited in this direction.

The equation for the temperature we have already obtained the form

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-\lambda v.$$

The solution will be periodic with the same period T as t of the temperature at $x=0$, and it may thus be supposed b up of terms

$$P\cos n\omega t+Q\sin n\omega t,$$

where

$$\omega=\frac{2\pi}{T}.$$

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* Hagström, *Stockholm, Vet.-Ak. Ofvers.*, 48, 1891.

† See § 35.

The quantities P and Q will be functions of x which satisfy

$$\frac{d^2P}{dx^2}-\frac{\lambda}{\kappa}P=\frac{n\omega}{\kappa}Q,$$

$$\frac{d^2Q}{dx^2}-\frac{\lambda}{\kappa}Q=-\frac{n\omega}{\kappa}P,$$

since these results follow from equating the coefficients of $\sin n\omega t$ and $\cos n\omega t$ to zero in the temperature equation.

Thus we have $$\left(\frac{d^2}{dx^2}-a^2\right)^2P+\beta^4P=0,$$

where $$a^2=\frac{\lambda}{\kappa} \quad\text{and}\quad \beta^2=\frac{n\omega}{\kappa}.$$

Therefore $P=Ae^{-g_nx}\cos(g_n'x-\epsilon)+A'e^{g_nx}\cos(g_n'x-\epsilon')$,
where

$$g_n=\sqrt{\left\{\frac{a^2+\sqrt{(a^4+\beta^4)}}{2}\right\}},\quad g_n'=\sqrt{\left\{\frac{-a^2+\sqrt{(a^4+\beta^4)}}{2}\right\}},$$

and A, A', ϵ, ϵ' are arbitrary constants.

Since P vanishes when $x=\infty$, it follows that $A'=0$, and our equation becomes $$P=Ae^{-g_nx}\cos(g_n'x-\epsilon),$$
from which we obtain

$$Q=Ae^{-g_nx}\sin(g_n'x-\epsilon).$$

Thus the term $$P\cos n\omega t+Q\sin n\omega t$$

becomes $$Ae^{-g_nx}\cos(n\omega t-g_n'x+\epsilon),$$

and
$$\begin{aligned} v=A_0e^{-g_0x}&+A_1e^{-g_1x}\cos(\omega t-g_1'x+\epsilon_1)\\ &+A_2e^{-g_2x}\cos(2\omega t-g_2'x+\epsilon_2),\\ &+\dots, \end{aligned}$$

with the same notation as above.

It is clear that $$g_0=\sqrt{(\lambda/\kappa)}$$
and that the mean temperature is given by

$$A_0e^{-\sqrt{(\lambda/\kappa)}x},$$

while
$$g_n=\sqrt{\left\{\frac{\lambda}{2\kappa}+\sqrt{\left(\frac{\lambda^2}{4\kappa^2}+\frac{n^2\omega^2}{4\kappa^2}\right)}\right\}},$$

$$g_n'=\sqrt{\left\{-\frac{\lambda}{2\kappa}+\sqrt{\left(\frac{\lambda^2}{4\kappa^2}+\frac{n^2\omega^2}{4\kappa^2}\right)}\right\}},$$

giving $$g_ng_n'=\sqrt{\left(\frac{n^2\omega^2}{4\kappa^2}\right)}=\frac{n\pi}{\kappa T}.$$

In Ångström's experiments the heating and cooling effects we carried out for intervals of 12 minutes each. The period of th oscillations of the temperature in the rod was thus 24 minute The temperature at a fixed point was then taken after the lap of a considerable time from the beginning of the experiment, intervals of 1 minute, and in this way the temperature curve that point obtained. This curve should be periodic and of th same period.

By analysing this curve we may obtain the coefficients in th expression for the temperature, written in the form,

$$B_0+B_1\cos(\omega t+\beta_1)+B_2\cos(2\omega t+\beta_2)+\dots.$$

Similar observations for another point give the coefficients in t expression

$$C_0+C_1\cos(\omega t+\gamma_1)+C_2\cos(2\omega t+\gamma_2)+\dots$$

for the temperature there.

Comparing these with the expression for v, namely,

$$v=A_0e^{-g_0x}+A_1e^{-g_1x}\cos(\omega t-g_1'x+\epsilon_1)+\dots,$$

we see that $$\frac{B_1}{C_1}=\frac{A_1e^{-g_1x_1}}{A_1e^{-g_1x_2}}=e^{g_1(x_2-x_1)}$$

and that $$\beta_1-\gamma_1=g_1'(x_2-x_1).$$

If the distance between the points be l, we obtain from t formula

$$g_ng_n'=\frac{n\pi}{\kappa T},$$

the result $$\frac{\log B_1-\log C_1}{l}\left(\frac{\beta_1-\gamma_1}{l}\right)=\frac{\pi}{\kappa T}.$$

Therefore $$\kappa=\frac{\pi l^2}{T(\beta_1-\gamma_1)(\log B_1-\log C_1)}.$$

The conductivity is thus determined independently of the emi vity. By altering the nature of the surface of the bar so tha changes, the values obtained for κ should not vary. Ångstr made such changes, and his results confirmed the values gi by his earlier experiments.

When κ is known, λ can be found at once.

22. Conductivity Experiments upon Bars (*continued*).
Variable Temperature.

In the mathematical problem, where the semi-infinite rod is initially at zero temperature and the end $x=0$ is kept at temperature unity, the temperature at time t is given by

$$v=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{(\kappa t)}}^{\infty} e^{-\xi^2}\,d\xi$$
$$=1-\frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}} e^{-\xi^2}\,d\xi. \quad \text{(Cf. § 18, Ex. 1.)}$$

It might appear that this solution would afford a means of determining κ, since from the observed temperature at any point x_1 at the time t_1 the table of values of the function

$$\Theta(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,d\xi$$

would give the value of $\quad x_1/2\sqrt{(\kappa t_1)}$,

and thus κ would be known. The difficulty in using this method depends upon the fact that the end of the bar, in this case $x=0$, is generally heated by a current of water at the given constant temperature. Now experiment has shown that it is not true that the end of the bar immediately attains the temperature of the fluid; and thus the mathematical statement of the conditions of the experiment can only be accepted as an approximation. However it has been shown to be a fair approximation, and is still accepted as a means of determining the thermal constants.*

Kirchhoff and Hansemann,† who first discussed this case, made the assumption that the temperature at $x=0$ would be given by $C+\phi(t)$, where C was a constant and $\phi(t)$ a function of the time which was to be taken indefinitely small. The value of C was to be determined by temperature observations in the immediate neighbourhood of the heated end, and was not assumed to be equal to the temperature of the fluid by means of which the heat was supplied.

Another method of treating the same problem has been developed, and a series of experiments devised and carried out in the Berlin Physikalische Institut has proved its power. The assumption of

* Cf. *Ann. Physik, Leipzig* (N.F.), 66, p. 207, 1898.

† Cf. *Ann. Physik, Leipzig* (N.F.), 9, p. 1, 1880; (N.F.), 13, p. 406, 1881.

a sudden change at $x=0$ to the temperature of the hea
is avoided by considering the alteration of the temper
the time at two points x_1 and x_2 in the bar. It is sho
solution of the equation of conduction may be obtained
give the observed temperatures at these two points. Tl
is then available for the evaluation of the conductivity.
conditions at the end $x=0$ are only used to obtain a suita
matical form for the solution. Two distinct lines of tre
followed. In one the approximate solution is derived
condition that at $x=0$, $v=1$; and then this solution is
make it suit the observed temperatures. In the second t
mate solution is derived from the condition that at $x=($
This is taken to represent the facts of the case when th
is heated, not by the flow of water, but by radiation fr
of platinum kept at white heat and supposed to con
end of the bar a constant supply of heat. For the deta
methods we must refer to the papers noted below.*

23. Semi-Infinite Solid. Initial Temperature Zero. Temperature $\phi(t)$.

We have seen in § 9 that, when the surface tempera
with the time, the solution may be deduced, by Duhamel'
from the case in which this temperature is constant.

Now, in the Semi-Infinite Solid, where v has to satisfy

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},$$

$$v=0 \text{ when } t=0,$$

and $$v=1 \text{ at } x=0,$$

the solution is given by

$$v=1-\frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}} e^{-\xi^2}d\xi$$

$$=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{(\kappa t)}}^{\infty} e^{-\xi^2}d\xi.$$

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* Grüneisen, *Ann. Physik, Leipzig* (4. F.), 3, p. 43, 1900; Giebe,
1903; *Verh. D. physik. Ges.*, p. 60, 1903; Hobson and Diesselhor
leitung," *Enc. d. math. Wiss.*, Bd. V., Tl. I., pp. 224-227, 1905.

Therefore, if $$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},$$

$$v=0 \text{ for } t=0,$$

and $$v=\phi(t) \text{ at } x=0,$$

the solution is given by

$$v=\int_0^t \phi(\lambda)\frac{\partial}{\partial t}F(x,\ t-\lambda)\,d\lambda,^*$$

where $$F(x,\ t-\lambda)=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{(\kappa(t-\lambda))}}^{\infty} e^{-\xi^2}d\xi.$$

In this case

$$\frac{\partial}{\partial t}F(x,\ t-\lambda)=-\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4\kappa(t-\lambda)}}\frac{\partial}{\partial t}\frac{x}{2\sqrt{(\kappa(t-\lambda))}}$$

$$=\frac{x}{2\sqrt{(\pi\kappa(t-\lambda)^3)}}e^{-\frac{x^2}{4\kappa(t-\lambda)}}.$$

Therefore the solution of our problem is

$$v=\frac{x}{2\sqrt{(\pi\kappa)}}\int_0^t \phi(\lambda)\frac{e^{-\frac{x^2}{4\kappa(t-\lambda)}}}{(t-\lambda)^{\frac{3}{2}}}\,d\lambda.$$

Putting $$\frac{x}{2\sqrt{(\kappa(t-\lambda))}}=\mu,$$

we have $$t-\lambda=\frac{x^2}{4\kappa\mu^2}$$

and $$v=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{(\kappa t)}}^{\infty}\phi\left(t-\frac{x^2}{4\kappa\mu^2}\right)e^{-\mu^2}\,d\mu.$$

In this form it is clear that our solution satisfies the differential equation and the initial and boundary conditions.

24. Semi-Infinite Solid. Surface Temperature a Harmonic Function of the Time.

If the surface temperature in the Semi-Infinite Solid bounded by the plane $x=0$ is given by $v=A\cos(\omega t-\epsilon)$, and the initial temperature is an arbitrary function $v=f(x)$, we may solve the problem by putting $v=u+w$, where

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2},$$

$$u=0 \text{ initially},$$

and $$u=A\cos(\omega t-\epsilon) \text{ at } x=0;$$

* Cf. § 9, III.

also
$$\frac{\partial w}{\partial t}=\kappa\frac{\partial^2 w}{\partial x^2},$$

$$w=f(x) \text{ initially,} \quad \text{and} \quad w=0 \text{ at } x=0.$$

The value of u we have found in § 23 to be

$$u=\frac{2A}{\sqrt{\pi}}\int_{x/2\sqrt{(\kappa t)}}^{\infty}\cos\left(\omega\left(t-\frac{x^2}{4\kappa\mu^2}\right)-\epsilon\right)e^{-\mu^2}d\mu$$
$$=\frac{2A}{\sqrt{\pi}}\cos(\omega t-\epsilon)\int_{x/2(\kappa t)}^{\infty}\cos\frac{\omega x^2}{4\kappa\mu^2}e^{-\mu^2}d\mu$$
$$+\frac{2A}{\sqrt{\pi}}\sin(\omega t-\epsilon)\int_{x/2(\kappa t)}^{\infty}\sin\frac{\omega x^2}{4\kappa\mu^2}e^{-\mu^2}d\mu. \quad \ldots\ldots$$

But it is known * that

$$\frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{\cos}{\sin}\frac{\omega x^2}{4\kappa\mu^2}e^{-\mu^2}d\mu=e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\frac{\cos}{\sin}\sqrt{\left(\frac{\omega}{2\kappa}\right)}x.$$

---

* Let $\quad u=\int_0^{\infty}e^{-x^2}\cos\frac{a^2}{2x^2}dx \quad$ and $\quad v=\int_0^{\infty}e^{-x^2}\sin\frac{a^2}{2x^2}dx.$

Then
$$\frac{du}{da}=-\int_0^{\infty}e^{-x^2}\frac{a}{x^2}\sin\frac{a^2}{2x^2}dx$$
$$=-\sqrt{2}\int_0^{\infty}e^{-\frac{a^2}{2x^2}}\sin x^2dx,$$

and
$$\frac{d^2u}{da^2}=\sqrt{2}\int_0^{\infty}e^{-\frac{a^2}{2x^2}}\frac{a}{x^2}\sin x^2dx$$
$$=2\int_0^{\infty}e^{-x^2}\sin\frac{a^2}{2x^2}dx.$$

Thus
$$\frac{d^2u}{da^2}=2v.$$

Similarly it can be shown that $\dfrac{d^2v}{da^2}=-2u.$

Therefore
$$\frac{d^4u}{da^4}+4u=0,$$

and
$$u=e^{-a}(A\cos a+B\sin a)+e^{a}(A'\cos a+B'\sin a).$$

Now $|u|<\int_0^{\infty}e^{-x^2}dx<\frac{1}{2}\sqrt{\pi}$ for every real a.

Therefore A' and B' must vanish.

Thus
$$u=e^{-a}(A\cos a+B\sin a)$$

and
$$v=e^{-a}(A\sin a-B\cos a).$$

But when $\quad a=0,\ u=\frac{1}{2}\sqrt{\pi}$ and $v=0$.

Therefore $\quad A=\frac{1}{2}\sqrt{\pi} \quad$ and $\quad B=0.$

Thus we have shown that

$$\int_0^{\infty}e^{-x^2}\cos\frac{a^2}{2x^2}dx=\frac{1}{2}\sqrt{\pi}e^{-a}\cos a$$
$$\int_0^{\infty}e^{-x^2}\sin\frac{a^2}{2x^2}dx=\frac{1}{2}\sqrt{\pi}e^{-a}\sin a.$$

The integrals u and v are continuous functions of a, and the differe the above argument are justifiable. (Cf. *F.S.*, §§ 84, 86.)

Therefore when t is so great that we can replace the integrals of (1) by those of (2), u is given by the equation

$$u = Ae^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t - \sqrt{\left(\frac{\omega}{2\kappa}\right)}x - \epsilon\right).$$

Further, we know from § 18 that

$$w = \frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^\infty f(x')\left(e^{-\frac{(x-x')^2}{4\kappa t}} - e^{-\frac{(x+x')^2}{4\kappa t}}\right)dx'$$

$$= \frac{2}{\pi}\int_0^\infty dx'\int f(x')\sin \alpha x \sin \alpha x' e^{-\kappa\alpha^2 t}d\alpha,$$

and, as t increases, this expression gets smaller and smaller. Thus when a sufficient time has passed to allow the distribution of temperature to become purely periodic, and the influence of the initial distribution has passed away, the temperature is given by

$$v = Ae^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t - \sqrt{\left(\frac{\omega}{2\kappa}\right)}x - \epsilon\right).$$

This result might have been obtained directly, as in the discussion of Angström's method. If we assume that sufficient time has passed to allow the temperature throughout the solid to become periodic, it must be given by terms of the type

$$P\cos\omega t + Q\sin\omega t,$$

where P and Q are functions of x only.

Then, substituting $\quad v = P\cos\omega t + Q\sin\omega t$

in the equation $\quad \dfrac{\partial v}{\partial t} = \kappa\dfrac{\partial^2 v}{\partial x^2},$

we have, on equating the coefficients of $\cos\omega t$ and $\sin\omega t$,

$$\frac{d^2P}{dx^2} - \frac{\omega}{\kappa}Q = 0,$$

$$\frac{d^2Q}{dx^2} + \frac{\omega}{\kappa}P = 0.$$

Put $\quad \omega/\kappa = \mu^2,$

and we have $\quad \dfrac{d^4P}{dx^4} + \mu^4 P = 0,$

which gives $\quad P = A'e^{-\mu x/\sqrt{2}}\cos(\mu x/\sqrt{2} + \epsilon') + A''e^{\mu x/\sqrt{2}}\cos(\mu x/\sqrt{2} + \epsilon''),$

where A', A'', ϵ', and ϵ'' are arbitrary constants to be determined by the initial and boundary conditions.

But when $x \to \infty$, P must not be infinite.

Therefore $\quad A'' = 0,$

and $\quad P = A'e^{-\mu x/\sqrt{2}}\cos(\mu x/\sqrt{2} + \epsilon').$

Also since $\quad Q = \dfrac{1}{\mu^2}\dfrac{d^2P}{dx^2},$

it follows that $\quad Q = A'e^{-\mu x/\sqrt{2}}\sin(\mu x/\sqrt{2} + \epsilon'),$

and $\quad v = A'e^{-\mu x/\sqrt{2}}\cos(\omega t - \mu x/\sqrt{2} - \epsilon').$

The conditions at $x=0$ show that

$$A'=A,$$
$$\epsilon'=\epsilon,$$

and
$$v=Ae^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x-\epsilon\right),$$

as before.

When the surface temperature is a periodic function of the time $\phi(t)$, of period $2\pi/\omega$, we can obtain the solution by using the Fourier Series for $\phi(t)$:

$$\phi(t)=a_0+(a_1\cos\omega t+b_1\sin\omega t)+(a_2\cos\omega t+b_2\sin\omega t)+\ldots$$
$$=A_0+A_1\cos(\omega t-\epsilon_1)+A_2\cos(2\omega t-\epsilon_2)+\ldots.$$

With this value of $\phi(t)$ we have, from the above discussion,

$$v=A_0+A_1e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x-\epsilon_1\right)$$
$$+A_2e^{-\sqrt{\left(\frac{2\omega}{2\kappa}\right)}x}\cos\left(2\omega t-\sqrt{\left(\frac{2\omega}{2\kappa}\right)}x-\epsilon_2\right)+\text{etc.}$$

25. Semi-Infinite Solid. Radiation at the Surface into a Medium at Zero Temperature. Initial Temperature Constant.*

When the initial temperature is constant and equal to v_0, the equations for v are as follows:

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},$$
$$v=v_0 \text{ when } t=0,$$
$$-\frac{\partial v}{\partial x}+hv=0 \text{ when } x=0.$$

Let
$$\phi=v-\frac{1}{h}\frac{\partial v}{\partial x}.$$

Then we have
$$\frac{\partial \phi}{\partial t}=\kappa\frac{\partial^2 \phi}{\partial x^2},$$
$$\phi=v_0 \text{ when } t=0,$$
$$\phi=0 \text{ when } x=0.$$

Therefore, from § 18,

$$\phi(x,t)=\frac{2v_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}\,du,$$

and it will be noticed that, when $x\to\infty$, $\phi(x,t)$ has the limit v_0.

* The case of initial temperature $f(x)$ is treated in § 83.

To determine v we have the equation

$$\frac{\partial v}{\partial x}-hv=-h\,\phi(x,t).$$

Thus $$v=Ce^{hx}-he^{hx}\int_{\infty}^{x}\phi(\xi,t)\,e^{-h\xi}\,d\xi,$$

on integrating this equation in the usual way.

Therefore $$v=Ce^{hx}+h\int_0^{\infty}\phi(x+\eta,t)\,e^{-h\eta}\,d\eta,$$

on putting $\xi=x+\eta$.

But as $x\to\infty$, $\phi(x,t)$ has the limit v_0, and as v must be finite, it follows that C must be zero.

Hence the solution of our problem is given by

$$\begin{aligned} v&=h\int_0^{\infty}\phi(x+\eta,t)\,e^{-h\eta}\,d\eta\\ &=\frac{2v_0h}{\sqrt{\pi}}\int_0^{\infty}e^{-h\eta}\left[\int_0^{(x+\eta)/2\sqrt{(\kappa t)}}e^{-u^2}\,du\right]d\eta.\end{aligned}$$

Therefore

$$\begin{aligned} v=&\frac{2v_0}{\sqrt{\pi}}\left[-e^{-h\eta}\int_0^{(x+\eta)/2\sqrt{(\kappa t)}}e^{-u^2}\,du\right]_0^{\infty}\\ &\qquad+\frac{2v_0}{\sqrt{\pi}}\int_0^{\infty}e^{-h\eta}\frac{d}{d\eta}\left(\int_0^{(x+\eta)/2\sqrt{(\kappa t)}}e^{-u^2}\,du\right)d\eta\\ =&\frac{2v_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}\,du+\frac{v_0}{\sqrt{(\pi\kappa t)}}\int_0^{\infty}e^{-h\eta-\frac{(x+\eta)^2}{4\kappa t}}\,d\eta\\ =&\frac{2v_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}\,du+\frac{v_0}{\sqrt{(\pi\kappa t)}}e^{hx+h^2\kappa t}\int_0^{\infty}e^{-\frac{(x+\eta+2h\kappa t)^2}{4\kappa t}}\,d\eta.\end{aligned}$$

In the second integral put

$$x+\eta+2h\kappa t=2\sqrt{(\kappa t)}\,u.$$

Then

$$\begin{aligned} v=&\frac{2v_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}du+\frac{2v_0}{\sqrt{\pi}}e^{hx+h^2\kappa t}\int_{(x+2h\kappa t)/2\sqrt{(\kappa t)}}^{\infty}e^{-u^2}du\\ =&\frac{2v_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}du+\frac{2v_0}{\sqrt{\pi}}e^{hx+h^2\kappa t}\left(\int_0^{\infty}e^{-u^2}\,du-\int_0^{(x+2h\kappa t)/2\sqrt{(\kappa t)}}e^{-u^2}\,du\right).\end{aligned}$$

Therefore

$$\frac{v}{v_0}=\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}\right)+e^{hx+h^2\kappa t}\left\{1-\Theta\left(\frac{x}{2\sqrt{(\kappa t)}}+h\sqrt{(\kappa t)}\right)\right\},$$

where, as in § 18, $$\Theta(x)=\frac{2}{\sqrt{\pi}}\int_0^{x}e^{-u^2}du.*$$

* Cf. Kirchhoff, *loc. cit.*, Bd. IV., pp. 25-27; Boussinesq, *Théorie analytique de la chaleur*, T. II., §§ 165-167, Paris, 1903; Weber-Riemann, *loc. cit.*, Bd. II., § 38.

Put $x=0$, and we have

$$\frac{v}{v_0}=e^{h^2\kappa t}\{1-\Theta(h\sqrt{(\kappa t)})\}$$

$$=\frac{2}{\sqrt{\pi}}e^{h^2\kappa t}\int_{h\sqrt{(\kappa t)}}^{\infty}e^{-u^2}du.$$

We proceed to find the surface temperature after a considerable time has passed since cooling began.

It can easily be proved, by repeated integration by parts, that

$$\int_x^{\infty}e^{-x^2}dx=\frac{e^{-x^2}}{2}\left(\frac{1}{x}-\frac{1}{2x^3}+\frac{1.3}{2^2}\frac{1}{x^5}\dots+(-1)^{n-1}\frac{1.3\dots2n-3}{2^{n-1}}\frac{1}{x^{2n-1}}\right)$$

$$+(-1)^n\frac{1.3\dots2n-1}{2^n}\int_x^{\infty}e^{-x^2}\frac{dx}{x^{2n}}.$$

This series does not converge, since the ratio of the n^{th} term to the $(n-1)^{\text{th}}$ does not remain less than unity, as n increases. However, if we take n terms of the series—the remainder—namely,

$$\frac{1.3\dots2n-1}{2^n}\int_x^{\infty}\frac{e^{-x^2}}{x^{2n}}dx,$$

is less than the n^{th} term, since

$$\int_x^{\infty}e^{-x^2}\frac{dx}{x^{2n}}<e^{-x^2}\int_x^{\infty}\frac{dx}{x^{2n}}.$$

We can thus stop at any term, and take the sum of the terms up to this term as an approximation for the function, the error being less in absolute value than the last term we have retained.

If in this way we take

$$v=\frac{2v_0}{\sqrt{\pi}}e^{h^2\kappa t}\left\{\frac{1}{2}e^{-h^2\kappa t}\left(\frac{1}{h\sqrt{(\kappa t)}}-\frac{1}{2(h\sqrt{(\kappa t)})^3}\right)\right\}$$

$$=\frac{v_0}{\sqrt{\pi}}\left(\frac{1}{h\sqrt{(\kappa t)}}-\frac{1}{2(h\sqrt{(\kappa t)})^3}\right),$$

and choose t so great that

$$\frac{v_0}{(h\sqrt{(\kappa t)})^3}<2\epsilon\sqrt{\pi},$$

where ϵ is any positive quantity taken as small as we please, the error in taking

$$\frac{v_0}{h\sqrt{(\pi\kappa t)}}$$

for the temperature at the surface will be less in value than ϵ.

26. Semi-Infinite Solid. Radiation at the Surface into a Medium at Temperature f(t). Initial Temperature Zero.

In this problem the temperature v has to satisfy

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},$$

$$-\frac{\partial v}{\partial x}+hv=hf(t) \text{ at } x=0,$$

$$v=0 \text{ when } t=0.$$

Proceeding as in the last article, put

$$\phi=v-\frac{1}{h}\frac{\partial v}{\partial x}.$$

Then we have the following equations to determine ϕ:

$$\frac{\partial \phi}{\partial t}=\kappa\frac{\partial^2 \phi}{\partial x^2},$$

$$\phi=f(t) \text{ at } x=0,$$

$$\phi=0 \text{ when } t=0.$$

These equations have already been discussed in § 23, and we have seen that

$$\phi=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{(\kappa t)}}^{\infty} f\left(t-\frac{x^2}{4\kappa\mu^2}\right)e^{-\mu^2}\,d\mu.$$

Hence, as in § 25,

$$v=\frac{2h}{\sqrt{\pi}}\int_0^{\infty} e^{-h\eta}\,d\eta\int_{(x+\eta)/2\sqrt{(\kappa t)}}^{\infty} f\left(t-\frac{(x+\eta)^2}{4\kappa\mu^2}\right)e^{-\mu^2}\,d\mu.$$

27. Terrestrial Temperature.

Observations of the temperature at points near the surface of the earth have been carried out at a large number of meteorological stations in different parts of the world for many years. These results have established the existence of two distinct phenomena of terrestrial temperature.

The first is that the variations of the surface temperature from the heat by day to the cold by night do not affect the temperatures of points at a depth of more than 3-4 feet, while the yearly changes from the cold of winter to the heat of summer may be observed up to a depth of 60-70 feet. Below that depth the temperature remains practically constant from day to day and is not subject to alterations due to the changes at the surface. In other words, the heat waves due to the changes of

the temperature at the surface die away before they penetrate to a depth of more than 60-70 feet, and the heat which is thus transferred to the earth oscillates in the upper crust, and while it proceeds inwards at certain seasons of the year, at others it ascends and radiates into space at the surface.

However, after we pass the limit at which the temperature is affected by these surface changes, and reach the depths at which it remains constant from day to day and year to year, there is a marked increase in the temperature as we descend. The temperatures observed at a great number of points at considerable depth and at many different stations leave no doubt upon this phenomenon. It has been observed near the equator as well as in the temperate zone, and although the rate of increase varies with different places and is much greater in the neighbourhood of active volcanoes or thermal springs, it may roughly be taken as about 1° F. for every 50 feet of descent at depths up to about one mile.* This rise of temperature was ascribed, both by Fourier and Laplace, to the high initial temperature of the earth, this supply of heat being gradually diffused outwards, and still to a great extent preserved at the centre of the earth. Such an assumption does not require that the rate of increase of temperature should be uniform as we continue to descend, and other physical phenomena show that the interior of the earth cannot be a mass of molten rock.

The periodic changes in the temperature near the surface have been used by writers from Fourier and Poisson onwards in the determination of the conductivity of the Earth, these determinations becoming of increasing value in recent years owing to the growth in the number of stations at which thermometric observations have been made.

Since these daily and annual variations of surface temperature are noticeable only at points comparatively near the Earth's surface, the problem may be simplified by neglecting the curvature of the Earth and supposing the surface to be the plane $x=0$, which is subjected to a periodic change of temperature. This problem has

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* Some authorities now regard the average as more nearly 1° F. for every 60 feet of descent, or even 1° F. for every 70 feet. Cf. Sollas, *The Age of the Earth and other Geological Studies*, 1905.

been discussed in § 24,* and the temperature at the depth x below the surface was found to be

$$A_0+A_1e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x-\epsilon_1\right)$$
$$+A_2e^{-\sqrt{\left(\frac{\omega}{\kappa}\right)}x}\cos\left(2\omega t-\sqrt{\left(\frac{\omega}{\kappa}\right)}x-\epsilon_2\right)+\dots,$$

when the surface temperature is

$$v=A_0+A_1\cos(\omega t-\epsilon_1)+A_2\cos(2\omega t-\epsilon_2)+\dots.$$

It is true that the temperature at the surface of the Earth depends not only upon the time, but upon the position of the place of observation, and that the constants A_0, A_1, ... will be functions of the position of these points; but, if a comparatively small portion of the surface is considered, the temperature over this may still be supposed dependent only on the time, and the general principles under which the periodical surface changes are transmitted into the interior and there die away will be fully illustrated by the solution in this form.

Thus the theory shows that each partial wave is propagated with unaltered period inwards; that the amplitudes of the waves of shorter period diminish more rapidly than those of greater period; and that they also have a more rapid alteration of phase: while, on the other hand, the velocity of their propagation is smaller in the ratio of the square roots of the periodic times. It follows that the periodical variation takes a simpler form as we descend, where the partial waves of smaller period become more rapidly negligible, so that after a certain depth the principal wave with the largest period and greatest amplitude will alone be found: while at a still greater depth this will also have become negligible and the temperature will have become constant. The depth at which the amplitude of the yearly variation is *e.g.* 0·1 will be about 19 times greater than that at which the corresponding amplitude for the daily variation will occur, since

$$x\sqrt{\left(\frac{\pi}{\kappa T}\right)}=x'\sqrt{\left(\frac{\pi}{\kappa T'}\right)}$$

gives the ratio of the depths and $T'=365T$. This result, it will be seen, agrees with the temperature observations which have shown that while the daily variation is not noticeable after a depth

* See also Boussinesq, *loc. cit.*, T. I., pp. 210-228; and papers in *Bul. sci. math.*, *Paris* (Ser. 2), 39, 1915.

of 3 or 4 feet, the annual variation may be traced to a depth of 60 or 70 feet.

These features of the problem were all noticed by Fourier and Poisson, to whom this discussion is due. The simplest application of the solution to the determination of the Conductivity of the Earth at different places on its surface will be found in Kelvin's paper on "The Reduction of Observations of Underground Temperature."* The substance of the Earth is taken to be a homogeneous mass of such rock as we have on the surface at the place of observation, and the values obtained for the conductivity are to a very considerable extent affected by the nature of the soil or rock in which the thermometers are imbedded. The data are the temperature observations at places on the same vertical and at different depths, these observations extending over a considerable number of years. In Kelvin's memeir, Forbes' Edinburgh Observations for a period of 18 years were employed.

These observations allow the mean temperature curve for a year to be drawn, and its harmonic components to be obtained. In this way we find for the depths at x_1 and x_2, the temperatures v_1 and v_2 in the form,

$$v_1 = A_0' + A_1' \cos\left(\frac{2\pi}{T}t - \epsilon_1'\right) + A_2' \cos\left(\frac{4\pi}{T}t - \epsilon_2'\right) + \dots,$$

$$v_2 = A_0'' + A_1'' \cos\left(\frac{2\pi}{T}t - \epsilon_1''\right) + A_2'' \cos\left(\frac{4\pi}{T}t - \epsilon_2''\right) + \dots.$$

But according to the solution of § 24, we have

$$\begin{aligned} v = A_0 &+ A_1 e^{-\sqrt{\left(\frac{\pi}{\kappa T}\right)}x} \cos\left(\frac{2\pi}{T}t - \sqrt{\left(\frac{\pi}{\kappa T}\right)}x - \epsilon_1\right) \\ &+ A_2 e^{-\sqrt{\left(\frac{2\pi}{\kappa T}\right)}x} \cos\left(\frac{4\pi}{T}t - \sqrt{\left(\frac{2\pi}{\kappa T}\right)}x - \epsilon_2\right) \\ &+ \dots. \end{aligned}$$

Therefore we should have

$$A_0' = A_0'',$$

$$A_n' = A_n e^{-\sqrt{\left(\frac{n\pi}{\kappa T}\right)}x_1},$$

$$A_n'' = A_n e^{-\sqrt{\left(\frac{n\pi}{\kappa T}\right)}x_2},$$

$$\epsilon_n' = \sqrt{\left(\frac{n\pi}{\kappa T}\right)}x_1 + \epsilon_n,$$

$$\epsilon_n'' = \sqrt{\left(\frac{n\pi}{\kappa T}\right)}x_2 + \epsilon_n.$$

* *Edinburgh, Trans. R. Soc.*, **22**, p. 405, 1861.

Thus $$\frac{\log A_n' - \log A_n''}{x_2 - x_1} = \frac{\epsilon_n'' - \epsilon_n'}{x_2 - x_1} = \sqrt{\left(\frac{n\pi}{\kappa T}\right)}.$$

The results of the calculations of the mean temperature curves at different depths give values for A_0', A_0'', ..., which vary only to a very slight extent. These agree with the theoretical result that the mean temperature due to the surface changes should not vary as we descend.

The first harmonic term, or the annual variation, is the largest, and observations based upon it will therefore be most trustworthy. Kelvin found that there was almost complete agreement between the values of
$$\frac{\log A_1' - \log A_1''}{x_2 - x_1}$$
and
$$\frac{\epsilon_1'' - \epsilon_1'}{x_2 - x_1},$$
the two expressions which should each be equal to $\sqrt{(\pi/\kappa)}$, the unit of time being the year.

From these results the value of κ, or $K/c\rho$, was obtained for the material at the place of observation.

Calculations were also made of the second harmonic amplitude and epoch. In this case the different temperature curves for the different years gave fundamental differences in the coefficients for the semi-annual period. These discrepancies and others in the case of the higher harmonics are not to be wondered at, as the actual state of affairs is not the ideal one which has been postulated with regard to the periodical variation of the temperature and the material of the Earth.

28. The Age of the Earth.

We have seen in § 27 that after the limits at which the temperature is affected by the surface changes are passed, a marked increase is observed as the depth increases, and that this temperature gradient has been taken, in ordinary circumstances, as about 1° F. for every 50 feet of descent up to a depth of about one mile. That this gradient might be used to obtain a rough estimate of the time that has elapsed since the Earth began to cool from its molten state, was remarked by Fourier himself.*

In the problem, as simplified by him for mathematical treatment,

* "Extrait d'un Mémoire sur le refroidissement séculaire du globe terrestre," *Bull. des sciences par la Société philomathique de Paris*, 1820. Also *Œuvres de Fourier*, T. II. (cf. p. 284).

the curvature of the Earth is neglected and the conductivity (κ) supposed constant. The surface is taken as the plane $x=0$, and radiation takes place into a medium at temperature zero; the temperature when cooling began—taken as the time $t=0$—is constant and equal to v_0. He obtained the result given in § 25 that for large values of t the temperature gradient near the surface is approximately $v_0/\sqrt{(\pi\kappa t)}$.

Kelvin * took the simpler problem of the semi-infinite solid bounded by $x=0$, the boundary being kept at zero temperature, the initial temperature being constant and equal to v_0. We have seen in § 18 that the temperature at the depth x at the time t is given by

$$v=\frac{2v_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}du.$$

Hence
$$\frac{\partial v}{\partial x}=\frac{v_0}{\sqrt{(\pi\kappa t)}}e^{-\frac{x^2}{4\kappa t}}.$$

When x is small and t is large, this becomes approximately

$$\frac{v_0}{\sqrt{(\pi\kappa t)}},$$

as in Fourier's problem.

With the value of κ used in Kelvin's paper (cf. *loc. cit.*, § 15)—namely 400—the units of length and time being the foot and year,† we have
$$\frac{\partial v}{\partial x}=\frac{v_0}{35{\cdot}4}\frac{1}{\sqrt{t}}e^{-\frac{x^2}{1600t}}.$$

Taking $v=7000°$ F. as a suitable temperature for melting rock, and $t=10^8$, we have
$$\frac{\partial v}{\partial x}=\frac{1}{50{\cdot}6}e^{-\frac{x^2}{16\times10^{10}}}.$$

Thus at $x=0$, the rate of increase of temperature is 1 in 50·6, and this temperature gradient will hold for about the first 10^5 feet.

At a depth of 4×10^5 feet, we obtain
$$\frac{\partial v}{\partial x}=\frac{1}{50{\cdot}6}e^{-1},$$

or about $\frac{1}{138}$ of a degree per foot; while at 8×10^5 feet we have
$$\frac{\partial v}{\partial x}=\frac{1}{50{\cdot}6}e^{-4},$$

or about $\frac{1}{2770}$ of a degree per foot.

Since the temperature gradients are inversely proportional to

* "The Secular Cooling of the Earth," *Edinburgh, Trans. R. Soc.*, **23**, p. 157, 1864.

† With c.g.s. units this value of κ will be ·0118. Cf. p. 8.

the square roots of the times, if the gradient is $\frac{1}{50}$ at 10^8 years, it will be $\frac{1}{100}$ at the same depth at 4×10^8 years; while gradients of $\frac{1}{2}$, 1, and 2 per foot correspond to 160,000, 40,000, and 10,000 years respectively. We are thus led to the result that, with the approximation which our statement of the problem affords, for the last 96,000,000 years the rate of increase of temperature underground has gradually diminished from about $\frac{1}{10}$ of a degree Fahrenheit per foot to about $\frac{1}{50}$, and that the time which has been required for the transition from a melting state to that in which the present gradient holds will be 10^8 years. The assumption of a higher initial temperature, 10,000° F., an extremely high estimate, would increase the term required to 200,000,000 years. Even allowing for effects of higher temperature in altering the conductivities and specific heats of rocks, Kelvin held that this investigation justified the statement that the consolidation of the Earth, and the time from which cooling commenced, could not have taken place less than 20 million of years ago, or we should now have a more rapid increase of temperature as we descend, nor could it have taken place more than 400 million years ago, or we should not have so much as is required for the smallest value obtained at present from the temperature observations.

This assumption of temperatures of from 7000° F. to 10,000° F. he recognised* to be a high estimate for the temperature of molten rock, but he adopted it, as he was most anxious not to under-estimate the Age of the Earth, and his wish was to give the largest possible limits rather than the smallest. Later experiments upon the behaviour of rocks under high temperatures led him to believe that these temperatures are much higher than those required for a typical basalt of the primitive character, and that 1200° C. would be a fairer estimate. This change from 7000° F. to 1200° C. would reduce his estimate of 10^8 years to a little less than 10^7, and he seems to have been somewhat of the opinion of King† that we have no warrant in this argument for extending the Earth's age beyond 24 million of years.

The limits of the Age of the Earth given by Kelvin in 1864 attracted much attention, for the geologists then, as now, demanded

* Cf. *Nature*, 59, p. 438, 1895; also *Phil. Mag.*, *London* (Ser. 5), 47, p. 66, 1899.

† *Amer. J. Sci.*, *Newhaven, Conn.*, 45, 1893.

a much longer period of time for the cooling from the molten state, their arguments being based on the visible processes and effects of stratification. Since Kelvin's pronouncement much discussion has taken place between the physicists on the one hand, with the estimates based on the temperature gradient as only one of their methods of attacking the problem, and the geologists on the other.* And as Kelvin continued to attach much weight to the estimates from the observed temperature gradient, the simple mathematical problem treated above has become classical. However the discovery of radioactivity towards the beginning of the twentieth century has not only afforded new methods of attacking the problem of the Age of the Earth; it may be said to have definitely closed the controversy as to the reliability or otherwise of the results obtained by Kelvin's and other allied methods.† His paper "On the Secular Cooling of the Earth" has now only a historical interest.

But it is somewhat surprising that physicists attached so much importance to conclusions where the assumptions made were so far reaching and significant. There can be no doubt that the mathematical solution of the heat problems involved, based on hypotheses just as credible, even before the discovery of radioactivity, would have given results widely different from those which the geologists were urged to accept as the only answer to the question at issue.‡

* Cf. Woodward (i) "The Mathematical Theories of the Earth," *American Association for the Advancement of Science (Toronto)*, 1889; (ii) "The Century's Progress in Applied Mathematics," *Bull. Amer. Math. Soc.*, 6, p. 147, 1900.

† See Rutherford's works, *Radio-Activity* (2nd Ed.), § 271, 1905, and *Radioactive Substances and their Radiations*, §§ 258-260, 1913; also a little volume in Harper's Library of Living Thought, entitled *The Age of the Earth* by A. Holmes (1913). At the British Association Meeting in Edinburgh in 1921 a discussion on "The Age of the Earth" was opened by Lord Rayleigh, who, as R. J. Strutt, did most important work on the amount of radium in the earth's crust and its internal heat. The abstract of his address and the contributions by J. W. Gregory and Eddington to the discussion will be found in the *British Association Reports* (Edinburgh), 1921. Rayleigh's conclusion is that "radioactive methods of estimation indicate a moderate multiple of 1,000 million years as the possible and probable duration of the earth's crust as suitable for the habitation of living beings, and that no other considerations from the side of physics or astronomy afford any definite presumption against this estimate." His address in full appears in *Nature*, No. 2713, October 27th, 1921.

‡ There is an interesting series of papers by Perry in *Nature*, 51, 1895, the aim of which was to show that other possible internal conditions would give enormously greater ages than physicists had been inclined to allow. This was before the radioactive properties of minerals entered into the discussion. Heaviside also made important contributions to the discussion. (Cf. *Electromagnetic Theory*, Vol. II., Ch. V., entitled, *Mathematics and the Age of the Earth*, 1899. To some of these questions we shall return in Chapter XI. (Cf. §§ 99, 109.)

CHAPTER IV

LINEAR FLOW OF HEAT. SOLID BOUNDED BY TWO PARALLEL PLANES. FINITE ROD

ıtroductory.

ıe last chapter we have examined the different cases of Flow of Heat when the solid is bounded by the plane $x=0$ ınbounded in the direction of x positive, and we have seen e problems of the Semi-Infinite Rod are reduced to the ion of the same fundamental differential equation. In this we shall examine the corresponding problems when the x is limited to the interval $0 \leqq x \leqq l$, and we shall also see mathematical discussion of these problems may be used evaluation of the Conductivity and Emissivity. In the l work of the laboratory it is always possible to have the so great a length that, when they are heated at one end only, er end remains unaffected by the change of temperature the application of the source of heat, and the rod may be in the mathematical statement of the problem as unlimited direction. Yet some of the most trustworthy methods of ıg the conductivity are founded upon the mathematical on of the flow of heat in a bar of short length, both ends ıbjected to definite conditions of temperature.

inite Rod. Ends at Zero Temperature. Initial Tempera-). No Radiation at the Surface.

he origin be taken at one end of the rod, and let the length od be l.

the problem is reduced to the solution of the equations

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}, \quad (0 < x < l) \qquad (1)$$

$$v=0, \text{ when } x=0 \text{ and } x=l, \qquad (2)$$

$$v=f(x), \text{ when } t=0. \qquad (3)$$

If the initial distribution were

$$v = A_n \sin \frac{n\pi}{l} x,$$

it is clear that $\qquad v = A_n \sin \frac{n\pi}{l} x \, e^{-\kappa \frac{n^2\pi^2}{l^2} t}$

would satisfy all the conditions of the problem.

Let us suppose that the initial temperature, $f(x)$, is a bounded function satisfying Dirichlet's Conditions (*F.S.*, § 93) in the interval $(0, l)$.

Consider the function v defined by the infinite series

$$v = \sum_1^\infty a_n \sin \frac{n\pi}{l} x \, e^{-\kappa \frac{n^2\pi^2}{l^2} t}, \qquad (4)$$

where $\qquad a_n = \frac{2}{l} \int_0^l f(x') \sin \frac{n\pi}{l} x' \, dx'.$

This series, owing to the presence of the convergency factor $e^{-\kappa \frac{n^2\pi^2}{l^2} t}$, is uniformly convergent for any interval of x, when $t > 0$; and, regarded as a function of t, it is uniformly convergent when $t \geqq t_0 > 0$, t_0 being any positive number. (Cf. § 12.)

Thus the function v, defined by the series (4), is a continuous function of x, and a continuous function of t, in these intervals.*

It is easy to show that the series obtained by term by term differentiation of (4) with respect to x and t are also uniformly convergent in these intervals of x and t respectively. Thus they are equal to the differential coefficients of the function v.

Hence $\qquad \frac{\partial v}{\partial t} = -\sum_1^\infty \kappa \frac{n^2\pi^2}{l^2} a_n \sin \frac{n\pi}{l} x \, e^{-\kappa \frac{n^2\pi^2}{l^2} t}$

and $\qquad \kappa \frac{\partial^2 v}{\partial x^2} = -\sum_1^\infty \kappa \frac{n^2\pi^2}{l^2} a_n \sin \frac{n\pi}{l} x \, e^{-\kappa \frac{n^2\pi^2}{l^2} t},$

when $t > 0$, and $0 < x < l$.

Thus the equation $\qquad \frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}$

is satisfied at all points of the rod, when $t > 0$, by the function defined by (4).

We have now to see whether this function also satisfies the Boundary and Initial Conditions.

* Regarded as a function of the two variables x, t, it is a continuous function of (x, t) in the regions $0 \leqq x \leqq l$, $t \geqq t_0 > 0$. (Cf. *F.S.*, § 37.)

Since the series is uniformly convergent with respect to x in the interval $0 \leqq x \leqq l$, when $t > 0$, it represents a continuous function of x in this interval.

Thus

$$\underset{x \to 0}{\mathrm{Lt}}\, v = \text{the value of the sum of the series when } x = 0$$
$$= 0,$$

and

$$\underset{x \to l}{\mathrm{Lt}}\, v = \text{the value of the sum of the series when } x = l$$
$$= 0.$$

Hence the *Boundary Conditions* are satisfied.

With regard to the *Initial Conditions*, we may again use the extension of Abel's Theorem contained in *F.S.*, § 73 I.

We have assumed that $f(x)$ is bounded and satisfies Dirichlet's Conditions in $(0, l)$.

Therefore the Sine Series for $f(x)$,

$$a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \ldots,$$

converges, and its sum is $f(x)$ at every point between 0 and l where $f(x)$ is continuous, and $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at all other points.* (Cf. *F.S.*, § 98.)

It follows from the extension of Abel's Theorem referred to above that when v is defined by (4), we have

$$\underset{t \to 0}{\mathrm{Lt}}\, v = \underset{t \to 0}{\mathrm{Lt}} \sum_1^\infty a_n \sin \frac{n\pi}{l} x\, e^{-\kappa \frac{n^2\pi^2}{l^2} t}$$
$$= f(x) \text{ at a point of continuity}$$
$$= \tfrac{1}{2}\{f(x+0)+f(x-0)\} \text{ at all other points.}$$

Thus we have shown that if the initial temperature satisfies Dirichlet's Conditions, and is continuous from $x=0$ to $x=l$, while $f(0)=f(l)=0$, the function defined by (4) † satisfies all the conditions of the problem.

If the initial temperature has discontinuities, the function defined by (4) at these points tends to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ as $t \to 0$. If t

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* If $f(x)$ is bounded and satisfies Dirichlet's Conditions, it follows from *F.S.* § 93 that it can only have ordinary discontinuities.

† This can be written as

$$v = \frac{2}{l}\int_0^l f(x') \sum_1^\infty \left(\sin \frac{n\pi}{l} x' \sin \frac{n\pi}{l} x\, e^{-\kappa \frac{n^2\pi^2}{l^2} t}\right) dx',$$

since the series under the integral is uniformly convergent. (*F.S.*, § 70.)

is taken small enough, v will bridge the gap from $f(x-0)$ to $f(x+0)$, and the temperature curve will pass close to the point $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

It must be remembered that the physical problem, as we have stated it for discontinuity, either at the ends of the rod or in the rod itself, is an ideal one. In nature there cannot be a discontinuity in the temperature in the rod initially. In the physical problem we must assume that a sudden change of temperature takes place at the instant from which our observations are measured, in the immediate neighbourhood of the point of discontinuity or the ends, if they are points of discontinuity. The gap in the temperature is thus smoothed over. The solution of the mathematical problem we have obtained satisfies these conditions, and it may be taken as representing the physical problem in this modified aspect.

31. Some further remarks may be made as to the $\underset{t\to 0}{\mathrm{Lt}}\, v$, and the way in which the function $v(x, t)$ defined by (4) satisfies the Initial Conditions.

I. We know that the Sine Series for $f(x)$, under the conditions stated in the previous section, is uniformly convergent in an interval (a, β), if $f(x)$ is continuous in that interval and at its ends. (*F.S.*, § 107.)

It follows from *F.S.*, § 73, I., that $v(x,t)$ converges uniformly to $f(x)$ as $t\to 0$ in that interval. In other words, given the arbitrary positive number ϵ, there exists a positive number τ such that

$$|v(x,t)-f(x)|<\epsilon, \quad \text{when } 0<t\leqq\tau,$$

the same τ serving for all points in (a, β).

Let x_0 be a point within (a, β).

Then there is a positive number η such that

$$|f(x)-f(x_0)|<\epsilon, \quad \text{when } |x-x_0|\leqq\eta.$$

Take the rectangle in the (x, t) plane given by

$$0<t\leqq\tau, \quad x_0-\eta\leqq x\leqq x_0+\eta,$$

where
$$a\leqq x_0-\eta<x_0+\eta\leqq\beta.$$

Let (x, t) be any point of this rectangle.

Then
$$v(x,t)-f(x_0)=\{v(x,t)-f(x)\}+\{f(x)-f(x_0)\}.$$

Therefore
$$\begin{aligned}|v(x,t)-f(x_0)| &\leqq |v(x,t)-f(x)|+|f(x)-f(x_0)|\\ &< \epsilon + \epsilon\\ &< 2\epsilon, \quad \text{when } 0<t\leqq\tau.\end{aligned}$$

Thus $v(x,t)$ tends to $f(x_0)$, when the point (x,t) moves along any path in this rectangle towards $(x_0, 0)$.

On the other hand, if x_0 is a point in $0<x<l$ at which $f(x_0+0)$ and $f(x_0-0)$ exist, and are different from $f(x_0)$, all that can be said is that $v(x,$

tends to $\frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$ as $t\to 0$, when the point (x, t) moves along the line $x=x_0$ towards the point (x_0, t).

II. In the argument of § 30 it is assumed that $f(x)$ is bounded and satisfies Dirichlet's Conditions in $(0, l)$, so that we can replace it by the Sine Series. But our physical intuition tells us that there must be a solution for our problem corresponding to any conceivable initial distribution of temperature in the rod, and, in particular, for any distribution which is continuous. However a continuous function need not satisfy Dirichlet's Conditions, and in fact it is known that there are continuous functions whose Fourier's Series diverge at an infinite number of points of the given interval.*

Fejér's Theorem (*F.S.*, § 101) and Bromwich's Theorem (*F.S.*, § 73, II.) furnish the mathematical demonstration that the series (4) is the solution of our problem, when all that is assumed as to the initial temperature $f(x)$ is that it is bounded and integrable in the interval $0 \leqq x \leqq l$.†

The proof does not differ much from that given above.

Since $f(x)$ is bounded and integrable and

$$a_n = \frac{2}{l}\int_0^l f(x')\sin\frac{n\pi}{l}x'\,dx',$$

it is clear that $a_n < M$ for every positive integer n, M being some positive number independent of x and n.

Also the series
$$\sum_1^\infty a_n \sin\frac{n\pi}{l}x\, e^{-\kappa\frac{n^2\pi^2}{l^2}t},$$

and the various series obtained by term by term differentiation with regard to x or t, converge uniformly through the region $0 \leqq x \leqq l$, $t \geqq t_0 > 0$, when t_0 is an arbitrary positive number.

Thus the differential equation and the boundary conditions are satisfied, as before.

Further, by Fejér's Theorem, the series of Arithmetic Means for the Sine Series for $f(x)$ converges to $f(x)$, if the function is continuous at that point.

Then, by Bromwich's Theorem, with the theorem given in *F.S.*, § 73, V.,

$$\underset{t\to 0}{\mathrm{Lt}}\, v(x, t) = f(x),$$

and $v(x, t)$ approaches the limit $f(x)$ uniformly as $t\to 0$, when x lies in an interval in which $f(x)$ is continuous.

If x is an ordinary point of discontinuity of $f(x)$, $f(x_0 \pm 0)$ existing but differing from $f(x)$, Fejér's Theorem and Bromwich's Theorem, show that

$$\underset{t\to 0}{\mathrm{Lt}}\, v(x, t) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

The remarks in I. as to the way in which $v(x, t)$ tends to its limit, apply also to these cases.

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* Cf. Fejér, *Ann. sci. Éc. norm., Paris* (Sér. 3), 28, p. 63, 1911.

† Cf. Moore, *Bull. Amer. Math. Soc.*, 25, p. 269, 1919.

32. Finite Rod. Radiation at the Surface. Ends at Fixed Temperatures.

Steady Temperature.

When the surface is not impervious to heat and the temperature of the medium is taken as zero, the equation for the temperature is

$$\frac{\partial v}{\partial t}=\frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2}-\frac{Hp}{c\rho\omega}v$$

with the notation of § 19.

The observation of the Steady Temperature in such a bar, when its ends are kept at constant temperatures V_1 and V_2, is one of the earliest methods of obtaining the relative values of the conductivities of different solids.

If we put $\mu^2=Hp/K\omega$, we have the equations

$$\frac{d^2v}{dx^2}-\mu^2 v=0, \quad (0<x<l)$$

$$v=V_1, \quad \text{when } x=0,$$

$$v=V_2, \quad \text{when } x=l,$$

and our solution is given by

$$\left.\begin{aligned} & v=Ae^{\mu x}+Be^{-\mu x}\\ \text{where}\quad & V_1=A\quad+B\\ \text{and}\quad & V_2=Ae^{\mu l}+Be^{-\mu l}.\end{aligned}\right\}$$

Thus
$$v=\frac{V_1\sinh\mu(l-x)+V_2\sinh\mu x}{\sinh\mu l}.$$

Let the temperatures be v_1, v_2 and v_3 at the points x_1, x_2 and x_3, where

$$x_3-x_2=x_2-x_1=a.$$

Then
$$\frac{v_1+v_3}{v_2}=2\cosh\mu a=2n, \text{ say.}$$

Hence
$$e^{\mu a}=n+\sqrt{(n^2-1)},$$

a result independent of V_1 and V_2. For two bars of the same perimeter, cross-section, and emissivity, it follows that

$$\sqrt{\left(\frac{K_1}{K_2}\right)}=\frac{\mu_2}{\mu_1}=\frac{\log(n_2+\sqrt{(n_2^2-1)})}{\log(n_1+\sqrt{(n_1^2-1)})}.*$$

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*Cf. § 13; also Preston, *loc. cit.*, §§ 296-299.

33. Finite Rod. Ends at Fixed Temperatures. Initial Temperature f(x). No Radiation at the Surface.

In this case we have the equations

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}, \quad (0<x<l)$$

$$v=v_1, \quad \text{when } x=0,$$

$$v=v_2, \quad \text{when } x=l,$$

and $$v=f(x), \text{ when } t=0.$$

As in § 9, we reduce this to a case of steady temperature, and a case where the ends are kept at zero temperature.

Put $$v=u+w,$$

where u and w satisfy the following equations:

$$\frac{d^2u}{dx^2}=0, \quad (0<x<l)$$

$$u=v_1, \text{ when } x=0,$$

$$u=v_2, \text{ when } x=l,$$

and $$\frac{\partial w}{\partial t}=\kappa\frac{\partial^2 w}{\partial x^2}, \quad (0<x<l)$$

$$w=0, \text{ when } x=0 \text{ and } x=l,$$

$$w=f(x)-u, \text{ when } t=0.$$

We find at once that $u=v_1+(v_2-v_1)x/l$,

and it follows from § 30 that

$$w=\sum_1^\infty a_n \sin\frac{n\pi}{l}x e^{-\kappa\frac{n^2\pi^2}{l^2}t},$$

where $$a_n=\frac{2}{l}\int_0^l\left[f(x')-\left(v_1+(v_2-v_1)\frac{x'}{l}\right)\right]\sin\frac{n\pi}{l}x'\,dx'.$$

Thus

generalization

$$v=v_1+(v_2-v_1)\frac{x}{l}+\frac{2}{\pi}\sum_1^\infty\frac{v_2\cos n\pi-v_1}{n}\sin\frac{n\pi}{l}x e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$

$$+\frac{2}{l}\sum_1^\infty\sin\frac{n\pi}{l}x e^{-\kappa\frac{n^2\pi^2}{l^2}t}\int_0^l f(x')\sin\frac{n\pi}{l}x'\,dx'.$$

When radiation takes place at the surface, the equations

$$\frac{\partial v}{\partial t}=\frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2}-\frac{Hp}{c\rho\omega}v, \quad (0<x<l)$$

$$v=v_1, \quad \text{when } x=0,$$

$$v=v_2, \quad \text{when } x=l,$$

and $$v=f(x), \text{ when } t=0,$$

are solved by substituting $$v=e^{-\nu t}u,$$

where $$\nu=Hp/c\rho\omega,$$

and then using the results of this and the next section.

34. Finite Rod. Ends at Temperatures $\phi_1(t)$ and $\phi_2(t)$. Initial Temperature f(x). No Radiation at the Surface.

In this case we have the equations

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}, \quad (0<x<l)$$

$$v=\phi_1(t), \text{ when } x=0,$$

$$v=\phi_2(t), \text{ when } x=l,$$

and $$v=f(x), \quad \text{when } t=0.$$

Following the general method given at the close of § 9, put

$$v=u+w,$$

where $$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2}, \quad (0<x<l)$$

$$u=0, \text{ when } x=0 \text{ and } x=l,$$

$$u=f(x), \text{ when } t=0,$$

and $$\frac{\partial w}{\partial t}=\kappa\frac{\partial^2 w}{\partial x^2}, \quad (0<x<l)$$

$$w=\phi_1(t), \text{ when } x=0,$$

$$w=\phi_2(t), \text{ when } x=l,$$

$$w=0, \text{ when } t=0.$$

The value of u follows from § 30, and is given by

$$u=\frac{2}{l}\sum_1^\infty e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x\int_0^l f(x')\sin\frac{n\pi}{l}x'\,dx'.$$

To obtain w we may use Duhamel's Theorem (§ 9), where the solution for the surface temperatures $\phi_1(t)$ and $\phi_2(t)$ is derived from that for the surface temperatures v_1 and v_2.

In this case the temperature at time t, when the temperature through the rod at $t=\lambda$ is zero, and the ends are kept at $\phi_1(\lambda)$ and $\phi_2(\lambda)$ from $t=\lambda$ to $t=t$, is given by

$$w=\phi_1(\lambda)\left[1-\frac{x}{l}-\frac{2}{\pi}\sum_1^\infty\frac{1}{n}e^{-\kappa\frac{n^2\pi^2}{l^2}(t-\lambda)}\sin\frac{n\pi}{l}x\right]$$
$$+\phi_2(\lambda)\left[\frac{x}{l}+\frac{2}{\pi}\sum_1^\infty\frac{1}{n}\cos n\pi e^{-\kappa\frac{n^2\pi^2}{l^2}(t-\lambda)}\sin\frac{n\pi}{l}x\right].$$

Hence, when the surface temperatures are $\phi_1(t)$ and $\phi_2(t)$, we obtain

$$w=\int_0^t\left[\phi_1(\lambda)\frac{\partial}{\partial t}F_1(x,\,t-\lambda)+\phi_2(\lambda)\frac{\partial}{\partial t}F_2(x,\,t-\lambda)\right]d\lambda,$$

where $\quad F_1(x,\,t-\lambda)=1-\dfrac{x}{l}-\dfrac{2}{\pi}\displaystyle\sum_1^\infty\frac{1}{n}e^{-\kappa\frac{n^2\pi^2}{l^2}(t-\lambda)}\sin\frac{n\pi}{l}x,$

$$F_2(x,\,t-\lambda)=\frac{x}{l}+\frac{2}{\pi}\sum_1^\infty\frac{1}{n}\cos n\pi e^{-\kappa\frac{n^2\pi^2}{l^2}(t-\lambda)}\sin\frac{n\pi}{l}x.$$

Thus

$$w=\frac{2\kappa\pi}{l^2}\sum_1^\infty ne^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x\int_0^t e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}(\phi_1(\lambda)-(-1)^n\phi_2(\lambda))\,d\lambda.$$

Therefore, finally,

$$v=\frac{2}{l}\sum_1^\infty e^{-\kappa\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x\left[\int_0^l f(x')\sin\frac{n\pi}{l}x'\,dx'\right.$$
$$\left.+\frac{n\kappa\pi}{l}\int_0^t e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}(\phi_1(\lambda)-(-1)^n\phi_2(\lambda))\,d\lambda\right].$$

This solution may also be obtained by the method used by Stokes in different Potential Problems.

If we assume that v can be expanded in Fourier's Sine Series,

$$\sum_1^\infty a_n\sin\frac{n\pi}{l}x,$$

a_n is a function of t given by

$$\frac{2}{l}\int_0^l v(x',\,t)\sin\frac{n\pi}{l}x'\,dx',$$

where $v(x,\,t)$ is the temperature at x at the time t.

Then, integrating by parts twice,

$$\int_0^l v(x',\,t)\sin\frac{n\pi}{l}x'\,dx'$$
$$=\frac{l}{n\pi}[v(0,\,t)-(-1)^n v(l,\,t)]-\frac{l^2}{n^2\pi^2}\int_0^l\frac{\partial^2 v}{\partial x'^2}\sin\frac{n\pi}{l}x'\,dx'.$$

But by our hypothesis,
$$\phi_1(t)=v(0,t),$$
$$\phi_2(t)=v(l,t),$$
and thus the coefficient of $\sin\frac{n\pi}{l}x$ in the expansion of $\frac{\partial^2 v}{\partial x^2}$, which is equal to
$$\frac{2}{l}\int_0^l \frac{\partial^2 v}{\partial x'^2}\sin\frac{n\pi}{l}x'\,dx',$$
is given by
$$-\frac{n^2\pi^2}{l^2}a_n+\frac{2n\pi}{l^2}(\phi_1(t)-(-1)^n\phi_2(t)).$$

Therefore we have to determine a_n from the equation
$$\frac{da_n}{dt}+\kappa\frac{n^2\pi^2}{l^2}a_n=\frac{2n\pi\kappa}{l^2}[\phi_1(t)-(-1)^n\phi_2(t)].$$

Therefore we have
$$a_n=C_n e^{-\kappa\frac{n^2\pi^2}{l^2}t}+\frac{2n\pi\kappa}{l^2}e^{-\kappa\frac{n^2\pi^2}{l^2}t}\int_0^t e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}(\phi_1(\lambda)-(-1)^n\phi_2(\lambda))\,d\lambda,$$
where C_n is a constant yet to be determined.

But, initially, $v=f(x)$.

Thus we must have
$$C_n=\frac{2}{l}\int_0^l f(x')\sin\frac{n\pi}{l}x'\,dx'.$$

Hence
$$v=\frac{2}{l}\sum_1^\infty \sin\frac{n\pi}{l}x\,e^{-\kappa\frac{n^2\pi^2}{l^2}t}\int_0^l f(x')\sin\frac{n\pi}{l}x'\,dx'$$
$$+\frac{2\kappa\pi}{l^2}\sum_1^\infty n\sin\frac{n\pi}{l}x\,e^{-\kappa\frac{n^2\pi^2}{l^2}t}\int_0^t e^{\kappa\frac{n^2\pi^2}{l^2}\lambda}(\phi_1(\lambda)-(-1)^n\phi_2(\lambda))\,d\lambda.*$$

35. Neumann's Bar Method of obtaining the Conductivity and Emissivity.

In his paper, *Über das Wärmeleitungsvermögen von Eisen und Neusilber*,† Weber describes a series of experiments which he conducted on a method suggested by Neumann in his lectures. The idea of this method is the same as in that of Ångström, but in this case both ends of the rod are subjected to periodical changes of temperature, so that the mathematical solution required is that of the preceding article. The end A of the rod AB is kept at temperature v_1, while B is kept at temperature v_2 for the interval $t=0$ to $t=T$. Then A is kept at v_2 and B at v_1 from $t=T$ to $t=2T$; and this is repeated indefinitely. When this series of surface temperatures has gone on for a sufficient time, the distribution of temperature in the bar approaches two limiting states, which continually repeat themselves, the one belonging to the even,

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* Cf. Mollison, *Mess. Math.*, *Cambridge*, 10, p. 170, 1881.

† *Ann. Physik, Leipzig*, 146, p. 257, 1872.

and the other to the odd period. Both of these are independent of the arbitrary initial distribution of temperature, and this may be taken as zero throughout the rod.

In §§ 33, 34 we have seen that the temperature at time t in a rod of length l whose ends are kept at $\phi_1(t)$ and $\phi_2(t)$, while the initial temperature of the rod is zero, and radiation takes place into a medium at zero temperature, is given by

$$v=\frac{2\pi\kappa}{l^2}\sum_1^\infty e^{-\left(\kappa\frac{n^2\pi^2}{l^2}+\nu\right)t} n\sin\frac{n\pi}{l}x\int_0^t e^{\left(\kappa\frac{n^2\pi^2}{l^2}+\nu\right)\lambda}(\phi_1(\lambda)-(-1)^n\phi_2(\lambda))\,d\lambda,$$

where $\kappa=K/c\rho$ and $\nu=Hp/c\rho\omega$,

In Neumann's Problem

$$\phi_1(t)=v_1 \text{ when } 2rT<t<(2r+1)T,$$
$$\phi_1(t)=v_2 \text{ when } (2r+1)T<t<(2r+2)T,$$

and

$$\phi_2(t)=v_2 \text{ when } 2rT<t<(2r+1)T,$$
$$\phi_2(t)=v_1 \text{ when } (2r+1)T<t<(2r+2)T,$$

r being zero or any positive integer.

Thus at the time $t=2rT+t'$, $(0<t'<T)$

$$v=\frac{2\kappa\pi}{l^2}\sum_1^\infty e^{-p_n t}n(v_1-(-1)^n v_2)\sin\frac{n\pi}{l}x$$
$$\left(\int_0^T e^{p_n\lambda}\,d\lambda-(-1)^n\int_T^{2T}e^{p_n\lambda}\,d\lambda+\dots\int_{2rT}^{2rT+t'}e^{p_n\lambda}\,d\lambda\right),$$

where $$p_n=\kappa\frac{n^2\pi^2}{l^2}+\nu.$$

Therefore $$v=\frac{2\kappa\pi}{l^2}\sum_1^\infty e^{-p_n t}\frac{n}{p_n}(v_1-(-1)^n v_2)\sin\frac{n\pi}{l}x$$
$$\left\{-1+(1+(-1)^n)\left(\frac{1-e^{2rp_nT}}{1+e^{-p_nT}}\right)+e^{p_n(2rT+t')}\right\}$$
$$=\frac{2\kappa\pi}{l^2}\sum_1^\infty\frac{n}{p_n}(v_1-(-1)^n v_2)\sin\frac{n\pi}{l}x$$
$$\left[-e^{-p_n t}+(1+(-1)^n)\frac{e^{-p_n t}-e^{-p_n t'}}{1+e^{-p_nT}}+1\right],$$

and in the limit, when t becomes very great, this expression for the value of v at the time t' in the even period becomes

$$v=\frac{2\kappa\pi}{l^2}\sum_1^\infty\frac{n}{p_n}(v_1-(-1)^n v_2)\sin\frac{n\pi}{l}x\left[1-(1+(-1)^n)\frac{e^{-p_n t'}}{1+e^{-p_nT}}\right].$$

In the odd period we obtain, in the same way, for the approximate value of v,

$$v=\frac{2\kappa\pi}{l^2}\sum_1^\infty\frac{n}{p_n}(v_1-(-1)^n v_2)\sin\frac{n\pi}{l}x\left[(1+(-1)^n)\frac{e^{-p_n t'}}{1+e^{-p_n T}}+(-1)^{n+1}\right]$$

Thus at the time t from the commencement of one of the even periods,

$$v=\frac{4\kappa\pi}{l^2}(v_1-v_2)\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x$$
$$+\frac{2\kappa\pi}{l^2}(v_1+v_2)\sum_1^\infty\frac{2n+1}{p_{2n+1}}\sin\frac{(2n+1)\pi}{l}x$$
$$-\frac{8\kappa\pi}{l^2}(v_1-v_2)\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x\frac{e^{-p_{2n}t}}{1+e^{-p_{2n}T}};$$

and at the time t from the commencement of one of the *odd* periods,

$$v=-\frac{4\kappa\pi}{l^2}(v_1-v_2)\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x$$
$$+\frac{2\kappa\pi}{l^2}(v_1+v_2)\sum_0^\infty\frac{2n+1}{p_{2n+1}}\sin\frac{(2n+1)\pi}{l}x$$
$$+\frac{8\kappa\pi}{l^2}(v_1-v_2)\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x\frac{e^{-p_{2n}t}}{1+e^{-p_{2n}T}},$$

where we have dropped the accent from t' and simplified the series.

These two expressions may be still further simplified, since

$$\frac{\sinh\mu x-\sinh\mu(l-x)}{2\sinh\mu l}=-\frac{4\pi}{l^2}\sum_1^\infty\frac{n}{\frac{4n^2\pi^2}{l^2}+\mu^2}\sin\frac{2n\pi}{l}x$$

and

$$\frac{\sinh\mu x+\sinh\mu(l-x)}{2\sinh\mu l}=\frac{2\pi}{l^2}\sum_0^\infty\frac{2n+1}{\frac{(2n+1)^2\pi^2}{l^2}+\mu^2}\sin\frac{(2n+1)\pi}{l}x.$$

Putting $\mu^2=\nu/\kappa$, we have

$$\frac{-\sinh\mu x+\sinh\mu(l-x)}{2\sinh\mu l}=\frac{4\kappa\pi}{l^2}\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x=U,\text{ say,}$$

$$\frac{\sinh\mu x+\sinh\mu(l-x)}{2\sinh\mu l}=\frac{2\kappa\pi}{l^2}\sum_0^\infty\frac{2n+1}{p_{2n+1}}\sin\frac{(2n+1)\pi}{l}x=V,\text{ say,}$$

since
$$p_{2n}=\kappa\frac{4n^2\pi^2}{l^2}+\nu$$

and
$$p_{2n+1}=\kappa\frac{(2n+1)^2\pi^2}{l^2}+\nu.$$

Therefore, for the *even* period, we have

$$v=(v_1-v_2)U+(v_1+v_2)V-\frac{8\kappa\pi}{l^2}(v_1-v_2)\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x\frac{e^{-p_{2n}t}}{1+e^{-p_{2n}T}},\qquad(1)$$

and for the *odd* period,

$$v=-(v_1-v_2)U+(v_1+v_2)V+\frac{8\kappa\pi}{l^2}(v_1-v_2)\sum_1^\infty\frac{n}{p_{2n}}\sin\frac{2n\pi}{l}x\frac{e^{-p_{2n}t}}{1+e^{-p_{2n}T}}.\qquad(2)$$

We shall now show how from the equations (1) and (2) we can obtain the values of H and K.

Since $\quad U=0$ at $x=\frac{1}{2}l$ and $V=\dfrac{1}{2\cosh\frac{1}{2}\mu l}$,

and the other terms vanish, the temperature $v_{\frac{1}{2}l}$, at the middle point of the bar, remains constant and equal to

$$\frac{v_1+v_2}{2\cosh\frac{1}{2}\mu l}.$$

Therefore $$\sqrt{\left(\frac{\nu}{\kappa}\right)}=\frac{2}{l}\log\left(a+\sqrt{(a^2-1)}\right),\qquad(3)$$

where $$2a=\frac{v_1+v_2}{v_{\frac{1}{2}l}}.$$

A simple way of obtaining another relation between the two unknown quantities is to take the difference of the temperatures at $x=l/6$ and $x=2l/3$ at any instant. For these points the terms in which n is a multiple of 2 or 3 disappear from the series in the expression for the difference of the temperatures, and this series is so rapidly convergent that we may neglect the term for $n=5$ and those which follow.

Thus, with this approximation, the difference of the temperatures at these two points at the time t after the beginning of one of the periods takes the form

$$M-Ne^{-pt},$$

where $$p=\kappa\frac{4\pi^2}{l^2}+\nu,$$

and M, N do not vary with t during the interval.

Let the differences of the temperatures at these points at the times t_1, t_2, $t_1+\beta$, and $t_2+\beta$ be d_1, d_2, d_1', and d_2'.

Then $$d_1-d_2=N(e^{-pt_2}-e^{-pt_1}),$$
$$d_1'-d_2'=N(e^{-pt_2}-e^{-pt_1})e^{-p\beta}.$$

Therefore $$e^{\nu\beta}=\frac{d_1-d_2}{d_1'-d_2'},$$

and $$\left(\kappa\frac{4\pi^2}{l^2}+\nu\right)\beta=\log(d_1-d_2)-\log(d_1'-d_2'). \qquad (4)$$

These two equations (3) and (4) are sufficient to determine K and H.

36. Finite Rod. Radiation at Ends into a Medium at Zero Temperature. Initial Temperature f(x). No Radiation at the Surface.

In this case the equations for the temperature are

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}, \quad (0<x<l), \qquad (1)$$

$$-\frac{\partial v}{\partial x}+hv=0 \quad \text{at} \quad x=0, \qquad (2)$$

$$\frac{\partial v}{\partial x}+hv=0 \quad \text{at} \quad x=l, \qquad (3)$$

and $$v=f(x), \text{ when } t=0. \qquad (4)$$

The expression $$e^{-\kappa a^2 t}(A\cos ax+B\sin ax)$$
satisfies (1).

It also satisfies (2) and (3), provided that

$$-aB+hA=0,$$

and $$a(B\cos al-A\sin al)+h(B\sin al+A\cos al)=0.$$

From these we obtain $$\frac{A}{a}=\frac{B}{h}$$

and $$\tan al=\frac{2ah}{a^2-h^2}. \qquad (5)$$

Hence the expression

$$A\left(\cos ax+\frac{h}{a}\sin ax\right)e^{-\kappa a^2 t}$$

satisfies (1), (2), and (3), where A is an arbitrary constant and a is any root other than zero of the equation

$$\tan al=\frac{2ha}{a^2-h^2}.$$

To form an idea of the distribution of the real roots of (5), it is only necessary to note that they correspond to the abscissae of the common points of the curves

$$\eta=\frac{2}{\tan\xi} \quad \text{and} \quad \eta=\frac{\xi}{hl}-\frac{hl}{\xi},$$

where we have put $al=\xi$.

The second of these curves is a rectangular hyperbola, whose centre is at the origin and whose asymptotes are

$$\xi=0 \quad \text{and} \quad \eta=\frac{\xi}{hl}.$$

If this hyperbola and the cotangent curve are drawn, it is clear from the figure that the positive roots lie one in each of the intervals $(0, \pi)$, $(\pi, 2\pi)$, ..., and the negative roots are equal in absolute value to the positive ones. Also there are no repeated roots.

Further, it is clear that (5) cannot have a pure imaginary root ib. Since we would have

$$\tanh lb+\frac{2hb}{b^2+h^2}=0,$$

which is impossible as both terms are of the same sign.

Also we shall see later* that it cannot have an imaginary root of the form $a\pm ib$; therefore its roots are all real.

Let us assume that $f(x)$ can be developed in an infinite series

$$f(x)=A_1X_1+A_2X_2+\ldots, \qquad (6)$$

where

$$\psi_n = X_n=\cos a_n x+\frac{h}{a_n}\sin a_n x,$$

a_n being the nth positive root of (5).

Then the solution of our problem is

$$v=\sum_1^\infty A_nX_ne^{-\kappa a_n^2 t}. \qquad (7)$$

The possibility of the expansion (6) and the question of the validity of this solution will be referred to again on p. 182, but if we assume that such an expansion exists, and that we may integrate the series term by term, the value of the coefficients may be obtained in a similar way to that in which the coefficients in Fourier's Series, with similar assumptions, may be found.

This depends upon the fact that

$$\int_0^l X_mX_n dx=0, \quad (m \neq n)$$

$$\int_0^l X_n^2 dx=\frac{(a_n^2+h^2)l+2h}{2a_n^2},$$

which we shall now prove.

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* Cf. p. 78.

Since $$\frac{d^2X_m}{dx^2}+a_m^2X_m=0,$$

and $$\frac{d^2X_n}{dx^2}+a_n^2X_n=0,$$ A

$$(a_m^2-a_n^2)\int_0^l X_mX_n\,dx=\int_0^l\left(X_m\frac{d^2X_n}{dx^2}-X_n\frac{d^2X_m}{dx^2}\right)dx$$

$$=\left[X_m\frac{dX_n}{dx}-X_n\frac{dX_m}{dx}\right]_0^l.$$

But $$-\frac{dX_r}{dx}+hX_r=0, \text{ when } x=0,$$

and $$\frac{dX_r}{dx}+hX_r=0, \text{ when } x=l,$$

whatever positive integer r may be.

Thus $$(a_m^2-a_n^2)\int_0^l X_mX_n\,dx=0,$$

and, when m is not equal to n,

$$\int_0^l X_mX_n\,dx=0.$$

To obtain the value of $\int_0^l X_n^2\,dx$, we note that

$$a_n^2\int_0^l X_n^2\,dx=-\int_0^l X_n\frac{d^2X_n}{dx^2}\,dx.$$ From A..

Thus $$a_n^2\int_0^l X_n^2\,dx=-\left[X_n\frac{dX_n}{dx}\right]_0^l+\int_0^l\left(\frac{dX_n}{dx}\right)^2dx$$

But $$a_nX_n=a_n\cos a_nx+h\sin a_nx,$$

and $$\frac{dX_n}{dx}=-a_n\sin a_nx+h\cos a_nx.$$

Therefore $$a_n^2X_n^2+\left(\frac{dX_n}{dx}\right)^2=a_n^2+h^2$$

and $$a_n^2\int_0^l X_n^2\,dx+\int_0^l\left(\frac{dX_n}{dx}\right)^2dx=(a_n^2+h^2)\,l.$$

But we have seen that

$$a_n^2\int_0^l X_n^2\,dx-\int_0^l\left(\frac{dX_n}{dx}\right)^2dx=-\left[X_n\frac{dX_n}{dx}\right]_0^l.$$

Therefore $$2a_n^2\int_0^l X_n^2\,dx = l(a_n^2+h^2) - \left[X_n\frac{dX_n}{dx}\right]_0^l.$$

But $$-\frac{dX_n}{dx}+hX_n=0, \text{ when } x=0,$$

and $$\frac{dX_n}{dx}+hX_n=0, \text{ when } x=l.$$

Therefore $$X_n\frac{dX_n}{dx}=-hX_n^2, \text{ when } x=l,$$

and $$X_n\frac{dX_n}{dx}= \quad hX_n^2, \text{ when } x=0.$$

But $$a_n^2X_n^2+\left(\frac{dX_n}{dx}\right)^2=a_n^2+h^2.$$

Therefore $$X_n^2=1, \text{ both when } x=0 \text{ and } x=l.$$

Thus $$\left[X_n\frac{dX_n}{dx}\right]_0^l=-2h$$

and $$\int_0^l X_n^2\,dx=\frac{(a_n^2+h^2)l+2h}{2a_n^2}.$$

Hence, if we assume the possibility of the expansion and that we may integrate term by term, we have

$$A_n\int_0^l X_n^2\,dx=\int_0^l f(x)X_n\,dx$$

and $$A_n=\frac{2a_n^2}{(a_n^2+h^2)l+2h}\int_0^l f(x)X_n\,dx.$$

Thus

$$v=2\sum_{n=1}^{\infty}e^{-\kappa a_n^2 t}\frac{a_n\cos a_nx+h\sin a_nx}{(a_n^2+h^2)l+2h}\int_0^l f(x)(a_n\cos a_nx+h\sin a_nx)\,dx.$$

We stated above (p. 75) that the equation

$$\tan al=\frac{2ah}{a^2-h^2}$$

cannot have an imaginary root of the form $a\pm ib$.

If this were possible, we would have two conjugate roots $a \pm ib$, and these would give the two expressions

$$X = \cos \alpha x + \frac{h}{\alpha} \sin \alpha x,$$

$$X' = \cos \alpha' x + \frac{h}{\alpha'} \sin \alpha' x,$$

where $\alpha = a + ib$

and $\alpha' = a - ib.$

Now we have seen that for any two unequal roots of (5),

$$\int_0^l X_m X_n \, dx = 0,$$

and this applies also to X, X', so that

$$\int_0^l XX' \, dx = 0.$$

But dividing X into its real and imaginary parts, we have

$$X = R + iS$$

and $X' = R - iS,$

so that we would have $\int_0^l (R^2 + S^2) \, dx = 0,$

which is impossible.

Thus we see that (5) has only real roots.

If radiation takes place at $x=0$ and $x=l$ into media at temperatures v_1 and v_2, the problem can be reduced to the above as usual by putting

$$v = u + w,$$

where u is a function of x only which satisfies the equations

$$\frac{d^2 u}{dx^2} = 0, \quad (0 < x < l)$$

$$-\frac{du}{dx} + h(u - v_1) = 0, \text{ when } x = 0,$$

and $$\frac{du}{dx} + h(u - v_2) = 0, \text{ when } x = l;$$

and w is a function of x and t which satisfies the equations

$$\frac{\partial w}{\partial t} = \kappa \frac{\partial^2 w}{\partial x^2}, \quad (0 < x < l)$$

$$-\frac{\partial w}{\partial x} + hw = 0, \text{ when } x = 0,$$

$$\frac{\partial w}{\partial x} + hw = 0, \text{ when } x = l,$$

and $$w = f(x) - u, \text{ when } t = 0.$$

The problems where one end of the rod is kept at a constant temperature, and radiation takes place at the other end, or when one end is rendered impervious to heat, may be treated in the same way.*

37. Application of this Solution to the Determination of the Conductivity and Emissivity.†

In the case of radiation at the surface of the rod into a medium at temperature zero, the solution may be deduced at once from that of § 36, and is given by

$$v=\sum_1^\infty A_n e^{-(\kappa a_n^2+\nu)t} X_n,$$

A_n and X_n having the values of that article, and ν being equal to $Hp/c\rho\omega$.

Neumann showed that this result may be used in determining the values of the thermal constants. His method requires the measurement of the temperatures v_0 and v_l, when $x=0$ and $x=l$.

Now $$X_n=\cos a_n x+\frac{h}{a_n}\sin a_n x,$$

and thus $X_n=1$, when $x=0$.

Also we have seen that $X_n^2=1$

when $x=l$. We proceed to determine the sign of X_n.

Since $$\tan a_n l=\frac{2a_n h}{a_n^2-h^2},$$

$$X_n=\left(\frac{h}{a_n}+\frac{a_n^2-h^2}{2a_n h}\right)\sin a_n l$$

$$=\frac{a_n^2+h^2}{2a_n h}\sin a_n l.$$

But $$0<a_1 l<\pi,$$

$$\pi<a_2 l<2\pi, \text{ etc.}$$

Hence $$X_n=(-1)^{n-1}, \text{ when } x=l.$$

Thus $$\tfrac{1}{2}(v_0+v_l)=A_1 e^{-\beta_1 t}+A_3 e^{-\beta_3 t}+\ldots,$$

and $$\tfrac{1}{2}(v_0-v_l)=A_2 e^{-\beta_2 t}+A_4 e^{-\beta_4 t}+\ldots,$$

where $$\beta_n=\kappa a_n^2+\nu.$$

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* Cf. also § 85.

† Neumann, *Ann. chim. phys., Paris* (Sér. 3), 66, p. 183, 1862.

As β_n increases with n, if t is chosen large enough, we shall obtain a close approximation by using only the first term in each of these series.

On this understanding

$$\tfrac{1}{2}(v_0+v_l)=A_1e^{-\beta_1 t},$$

and

$$\tfrac{1}{2}(v_0-v_l)=A_2e^{-\beta_2 t}.$$

In Neumann's experiment he first heated one end of the bar by a flame, and then allowed the bar to cool by radiation. After some time he began to take observations of $v_0 \pm v_l$ at equal intervals. These readings showed when the temperatures began to obey the law given above. By this means the constants β_1 and β_2 are found; and thus two equations are obtained from which the conductivity and emissivity may be determined. However, as the values of a_1 and a_2 involve h, this calculation has to proceed by successive approximations and is somewhat complicated.

A simpler method is obtained by observing also the temperature at the middle point of the bar.

When

$$x=\tfrac{1}{2}l,$$

$$X_n=\cos\tfrac{1}{2}a_nl+\frac{h}{a_n}\sin\tfrac{1}{2}a_nl.$$

But

$$\tan a_nl=\frac{2a_nh}{a_n^2-h^2}.$$

It follows that $\tan\frac{1}{2}a_nl$ is equal to h/a_n when n is odd, and to $-a_n/h$ when n is even.

Thus, for

$$x=\tfrac{1}{2}l,$$

$$X_n=\cos\tfrac{1}{2}a_nl\left(1+\frac{h}{a_n}\tan\tfrac{1}{2}a_nl\right)$$

$$=\frac{1}{\cos\frac{1}{2}a_nl},\text{ when } n \text{ is odd,}$$

$$=0,\text{ when } n \text{ is even.}$$

Thus

$$v_{\frac{1}{2}l}=A_1\frac{e^{-\beta_1 t}}{\cos\frac{1}{2}a_1l}+A_3\frac{e^{-\beta_3 t}}{\cos\frac{1}{2}a_3l}+\dots,$$

and to our approximation

$$\frac{v_0+v_l}{2v_{\frac{1}{2}l}}=\cos\tfrac{1}{2}a_1l.$$

From this result we find a_1; then h follows from the equation

$$\tan \tfrac{1}{2} a_1 l = \frac{h}{a_1},$$

and a_2 from

$$\tan \tfrac{1}{2} a_2 l = -\frac{a_2}{h}.$$

Also

$$\beta_1 = \kappa a_1^2 + \nu,$$
$$\beta_2 = \kappa a_2^2 + \nu$$

give

$$\kappa \text{ and } \nu,$$

so that the values of the conductivity K and the emissivity H may then be found.

38. Equation of Conduction in a Thin Wire heated by an Electric Current of Constant Strength.*

The equation for the temperature in a thin wire along which an electric current of constant strength is flowing was given by Verdet in 1872.† For some time little use was made of this method of heating the metal, although it has several obvious advantages. In the first place, the electrical measurements can be made with such accuracy that it is found possible to arrange the experiments so that the difference of temperature along the wire will be small. The error due to the neglect of the change of the Electrical Conductivity, and also, though not so marked, of the Thermal Conductivity, is thus avoided. Further, it is of importance that the same method of heating the wire should be employed in the cases when the temperatures to be examined are widely different, and that the two conductivities—electrical and thermal—should be obtained by simultaneous experiments. By using the equation of conduction in this form, the question raised by Wiedemann and Franz,‡ as to

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* Verdet, *Théorie Mécanique de la Chaleur*, T. II., p. 197, 1872.

† Alternating currents have also been used in this connection. Several important papers may be referred to:

Cranz, *Zs. Math., Leipzig*, 34, p. 92, 1889.
Ebeling, *Ann. Physik, Leipzig* (4. F.), 27, p. 391, 1908.
Weinreich, *Zs. Math., Leipzig*, 62, p. 1, 1914.

The last-named memoir contains a valuable account of the literature on this subject, and the variation of the temperature over the section as well as along the length of the wire is taken into consideration.

‡ *Ann. Physik, Leipzig*, 89, p. 497, 1853.

the ratio of the thermal and electrical conductivities, has been again examined, and it has been shown that the ratio is not so nearly constant as was at first supposed.

We shall first find the Equation of Conduction, and then show how the Steady Temperature and the Variable Temperature in such a wire have been used to determine the Electrical and Thermal Conductivity of the metal.

Let the wire be of length l, and let K, c, ρ, and H be its thermal conductivity, specific heat, density, and emissivity. Let i be the strength of the current, and σ the electrical conductivity, *i.e.* the reciprocal of the resistance per unit cross-section per unit length.

Consider the element of the wire contained between the sections distant x and $x+dx$ from one end.

The rate of gain of heat in this element from the flow of heat over the sections at x and $x+dx$ is ultimately

$$K\omega \frac{\partial^2 v}{dx^2} dx,$$

ω being the area of the cross-section of the wire.

The rate at which heat is lost at the surface of the element is

$$H(v-v_0)p\,dx,$$

p being the perimeter of the cross-section, and v_0 the temperature of the surrounding medium.

The rate of gain of heat due to the current i is

$$\frac{i^2}{\omega\sigma} dx.$$

The total rate of gain of heat is therefore

$$\left(K\omega\frac{\partial^2 v}{\partial x^2} - H(v-v_0)p + \frac{i^2}{\omega\sigma}\right)dx.$$

This must be equal to $\qquad \omega c\rho \dfrac{\partial v}{\partial t} dx,$

and therefore the equation of conduction is

$$\frac{\partial v}{\partial t} = \frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2} - \frac{Hp}{c\rho\omega}(v-v_0) + \frac{i^2}{c\rho\omega^2\sigma}.$$

Writing $\qquad \kappa = \dfrac{K}{c\rho}, \quad \lambda = \dfrac{Hp}{c\rho\omega}, \quad$ and $\quad a = \dfrac{i^2}{c\rho\omega^2\sigma},$

this equation becomes $\dfrac{\partial v}{\partial t} = \kappa\dfrac{\partial^2 v}{\partial x^2} - \lambda(v-v_0) + a.$

39. The Steady Temperature. Kohlrausch's Method of obtaining the ratio of the Electrical and Thermal Conductivities.

Kohlrausch* has shown how the Steady Temperature may be employed in finding the ratio of the electrical and thermal conductivities.

The ends of the wire are kept at as nearly as possible equal temperatures. The surface is supposed rendered impervious to heat, and the current i is supposed to have been flowing long enough to allow the steady rate of temperature to have been reached.

In this case the equation of conduction becomes

$$\frac{K}{c\rho}\frac{d^2v}{dx^2}+\frac{i^2}{c\rho\omega^2\sigma}=0$$

or

$$K\frac{d^2v}{dx^2}+\frac{i^2}{\sigma\omega^2}=0.$$

Let u be the potential at the section x.

Then
$$i=-\omega\sigma\frac{du}{dx}.$$

But
$$\frac{dv}{dx}=\frac{dv}{du}\frac{du}{dx}.$$

Therefore
$$\frac{dv}{dx}=-\frac{i}{\omega\sigma}\frac{dv}{du}$$

and
$$\frac{d^2v}{dx^2}=\frac{i^2}{\omega^2\sigma^2}\frac{d^2v}{du^2}.$$

Therefore we have

$$\frac{K}{\sigma}\frac{d^2v}{du^2}+1=0. \qquad (1)$$

Thus
$$\frac{K}{\sigma}v=-\frac{1}{2}u^2+Au+B, \qquad (2)$$

where A and B are constants determined by the temperatures at the ends.

Let (u_1, v_1) (u_2, v_2) and (u_3, v_3) be the values of u and v at any three sections x_1, x_2, x_3 of the wire. It follows from (2) that

$$\frac{2K}{\sigma}[v_1(u_2-u_3)+v_2(u_3-u_1)+v_3(u_1-u_2)]=(u_2-u_3)(u_3-u_1)(u_1-u_2). \qquad (3)$$

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* Kohlrausch, (1) *Berlin, SitzBer. Ak. Wiss.*, p. 714, 1899; (2) *Ann. Physik, Leipzig* (4 F.), 1, p. 132, 1900.

See also Czermak, *Wien, SitzBer. Ak. Wiss.*, 103 (II.a), p. 1107, 1894. Duncan, *Papers from the Department of Physics, No.* 11, *McGill University, Montreal*, 1900; and Weinreich, *loc. cit.*, p. 4.

When the temperatures at the ends of the wire are kept the sam the distribution of temperature in the wire will be symmetric about its middle point. Let the points x_1 and x_3 be at equal distanc from the middle point x_2 on either side.

Then $$v_1 = v_3$$

and $$u_1 - u_2 = u_2 - u_3.$$

Therefore we have from (3)

$$\frac{K}{\sigma}(v_2 - v_1) = \frac{1}{2}(u_1 - u_2)^2,$$

and we have thus obtained a simple method of determining t value of the ratio K/σ of the Thermal and Electrical Conductiviti involving only the reading of the difference of the temperatur and potentials at two points of the wire, when the current is regulated that the temperature of the wire is steady.

40. The Variable Temperature.

The variable temperature of a wire along which a consta electrical current is flowing, while radiation takes place at t surface, has also been used in determining the thermal and electri constants. The following investigation is due to Straneo.*

We have found the equation of conduction (§ 38) in the form

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} - \lambda (v - v_0) + a,$$

where $$\kappa = \frac{K}{c\rho}, \quad \lambda = \frac{Hp}{c\rho\omega}, \quad \text{and} \quad a = \frac{i^2}{c\rho\omega^2\sigma}.$$

Suppose the temperature of the medium into which radiati takes place to be zero, and that

$$v = 0, \text{ when } t = 0, \quad (0 < x < l)$$
$$v = 0, \text{ when } x = 0 \text{ and } x = l,$$

are the initial and boundary conditions.

To integrate the equation of conduction, we proceed as us to break up the problem into one of Steady Temperature and o of Variable Temperature.

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* Straneo, *Roma, Rend. Acc. Lincei* (Ser. 5), 7, Sem. ii., 1898.

See also Schaufelberger, *Ann. Physik, Leipzig*, (4. F.), 7, p. 589, 1902; Weinreich, *loc. cit.*

Put
$$v = u + w,$$
where u is independent of the time and satisfies the equations
$$\left.\begin{aligned} &\kappa \frac{d^2u}{dx^2} - \lambda u + a = 0 \\ &u = 0 \text{ at } x = 0 \text{ and } x = l \end{aligned}\right\},$$
and w is a function of x and t which satisfies the equations
$$\left.\begin{aligned} &\frac{\partial w}{\partial t} = \kappa \frac{\partial^2 w}{\partial x^2} - \lambda w \\ &w = 0 \text{ at } x = 0 \text{ and } x = l \\ &w = -u \text{ at } t = 0 \end{aligned}\right\}.$$

The value of u is obtained immediately in the form
$$u = b\left[1 - \frac{\sinh \mu x + \sinh \mu(l-x)}{\sinh \mu l}\right],$$
where
$$\mu = \sqrt{(\lambda/\kappa)} \quad \text{and} \quad b = a/\lambda.$$

This function may be expanded in the Sine Series
$$\sum_1^\infty A_n \sin \frac{n\pi}{l} x,$$
where $A_n = 0$, when n is an even integer, and
$$A_n = \frac{4b}{n\pi} \frac{l^2\mu^2}{n^2\pi^2 + l^2\mu^2},$$
when n is an odd integer.

With this value of u the solution of the equations for w follows immediately, and we have
$$w = -\sum_1^\infty A_n \sin \frac{n\pi}{l} x\, e^{-\left[\kappa \frac{n^2\pi^2}{l^2} + \lambda\right] t},$$
or
$$w = -\frac{4b}{\pi} \sum_{n=1}^\infty \frac{l^2\mu^2}{(2n-1)\{(2n-1)^2\pi^2 + l^2\mu^2\}} \sin \frac{(2n-1)\pi}{l} x\, e^{-\left[\kappa \frac{(2n-1)^2\pi^2}{l^2} + \lambda\right] t}.$$

Therefore
$$\begin{aligned} v = b&\left[1 - \frac{\sinh \mu x + \sinh \mu(l-x)}{\sinh \mu l}\right] \\ &- \frac{4b}{\pi} \sum_1^\infty \frac{l^2\mu^2}{(2n-1)\{(2n-1)^2\pi^2 + l^2\mu^2\}} \sin \frac{(2n-1)\pi}{l} x\, e^{-\left[\kappa \frac{(2n-1)^2\pi^2}{l^2} + \lambda\right] t}. \end{aligned}$$

In applying this solution we note that the coefficients of the terms in the series for w diminish rapidly, and when $x = l/3$ or $2l/3$ the

second term in w is zero. Hence to a close approximation the value of v at these points is given by

$$v=[u]_{\frac{1}{2}l}-\frac{2\sqrt{3}b}{\pi}\frac{l^2\mu^2}{\pi^2+l^2\mu^2}e^{-\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)t}.$$

Let observations be made at the point $x=\frac{1}{3}l$ at the times t_1, t_2 and t_3, where $t_2-t_1=t_3-t_2=\tau$.

Let the temperatures be v_1, v_2, and v_3 respectively.

Then
$$\log\frac{(v_2-v_1)}{(v_3-v_2)}=\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)\tau, \quad\text{.........(1)}$$

and
$$v_2-v_1=\frac{2b\sqrt{3}}{\pi}\frac{\mu^2l^2}{\pi^2+\mu^2l^2}\left(e^{-\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)t_1}-e^{-\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)t_2}\right). \quad\text{.........(2)}$$

Also take the value of the steady temperature at the middle point of the wire, viz.,

$$v_4=b\left(1-\frac{1}{\cosh\frac{1}{2}\mu l}\right). \quad\text{.........(3)}$$

These three equations determine the values of κ, λ, and σ. For from (1) we find the value of $\kappa\frac{\pi^2}{l^2}+\lambda$.

Inserting this in (2) we have an equation giving the value of μ, and with this value of μ from (3) we find b.

But
$$a=\frac{i^2}{c\rho\omega^2\sigma}, \quad\text{.........(4)}$$

$$b=\frac{a}{\lambda}, \quad\text{.........(5)}$$

$$\mu=\sqrt{(\lambda/\kappa)}. \quad\text{.........(6)}$$

Therefore, μ being known, we have from (1) and (6) the values of κ and λ, and b being known, the value of σ follows at once from (4) and (5).

In the actual experimental work this process was reversed. The wire was first heated by the current till it was observed that the steady temperature had been attained. The current was then cut off and the wire allowed to cool, the ends and the surrounding medium being still kept at the given constant temperature.

With this arrangement the steady temperature u is given as before by

$$u=b\left(1-\frac{\sinh\mu x+\sinh\mu(l-x)}{\sinh\mu l}\right),$$

and the variable temperature v is determined by the equations

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}-\lambda v,$$

$$v=0, \text{ when } x=0 \text{ and } x=l,$$

and $$v=u, \text{ when } t=0.$$

Thus

$$v=\frac{4b}{\pi}\sum_1^\infty\frac{l^2\mu^2}{(2n-1)\{(2n-1)^2\pi^2+l^2\mu^2\}}\sin\frac{(2n-1)\pi}{l}x\, e^{-\left(\kappa\frac{(2n-1)^2\pi^2}{l^2}+\lambda\right)t}.$$

Then, as before, we find

$$\log\frac{(v_1-v_2)}{(v_2-v_3)}=\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)\tau,$$

$$v_1-v_2=\frac{2b\sqrt{3}}{\pi^2+l^2\mu^2}\left(e^{-\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)t_1}-e^{-\left(\kappa\frac{\pi^2}{l^2}+\lambda\right)t_2}\right),$$

and $$v_4=b\left(1-\frac{1}{\cosh\frac{1}{2}\mu l}\right).$$

The values of κ, λ, and σ follow as above.

CHAPTER V

TWO-DIMENSIONAL PROBLEMS

41. Introductory.

In the last two chapters we have been examining different cases of Linear Flow of Heat. In these the temperature has been dependent only upon the time and upon one geometrical coordinate. Such problems may be referred to as one-dimensional. We proceed to the discussion of cases in which the flow of heat takes place in parallel planes. If these planes are taken parallel to the xy plane, the temperature will depend only upon x and y, if it is a case of steady temperature, or upon x, y, and t, if we are dealing with variable temperature. We speak of these problems as two-dimensional.

The first problem in the Conduction of Heat discussed in detail by Fourier in his treatise, is that of the Steady Temperature in the Infinite Solid bounded by the planes $x = \pm\frac{1}{2}\pi$, $y=0$, and extending to infinity in the direction y positive. The boundaries $x=\pm\frac{1}{2}\pi$ are kept at zero temperature, and the base $y=0$ at temperature unity. His discussion led him to the expansion of unity in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ in the series

$$\frac{4}{\pi}\{\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \dots\},^*$$

and he then proceeded to consider the question of the development of an arbitrary function in trigonometrical series, and obtained the expansion now known as Fourier's Series. He was thus able

* The series $\frac{4}{\pi}\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \dots\}$

may be obtained in the ordinary way as the Cosine Series for $f(x)$, when

$$f(x)=1, \qquad (0 < x < \tfrac{1}{2}\pi)$$
$$f(x)=-1. \qquad (\tfrac{1}{2}\pi < x < \pi)$$

to give the distribution of temperature in this solid, when the base is kept at the temperature $v=f(x)$, $f(x)$ being an arbitrary function of x, while the faces $x=\pm\frac{1}{2}\pi$ are kept as before at zero.

42. Infinite Rectangular Solid. Steady Temperature.

Instead of taking Fourier's Problem in the form which he adopted, we shall take the solid as bounded by the planes $x=0$ and $x=\pi$, which are kept at zero temperature, and the plane $y=0$, which is kept at the temperature $v=f(x)$. We assume that the function $f(x)$ is bounded and satisfies Dirichlet's Condition (*F.S.* § 93) in $(0, \pi)$.

The equations for the temperature will thus be as follows:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0, \quad (0<x<\pi, 0<y)$$

$$v=0, \text{ when } x=0 \text{ and } x=\pi,$$

and $$v=f(x), \text{ when } y=0. \quad (0<x<\pi)$$

Also we have $$\operatorname*{Lt}_{y\to\infty}(v)=0.$$

Starting with the Sine Series for $f(x)$,

$$a_1 \sin x+a_2 \sin 2x+\dots,$$

where $$a_n=\frac{2}{\pi}\int_0^\pi f(x')\sin nx'\,dx',$$

let us examine the function v defined by the equation

$$v=a_1e^{-y}\sin x+a_2e^{-2y}\sin 2x+\dots.$$

Since $f(x)$ is bounded and integrable in $(0, \pi)$ and

$$a_n=\frac{2}{\pi}\int_0^\pi f(x')\sin nx'\,dx',$$

then $$|a_n|<2M, \text{ where } |f(x)|<M \text{ in } (0, \pi).$$

Also $$|a_n \sin nx\, e^{-ny}|<2Me^{-ny_0}, \text{ when } y\geqq y_0>0,$$

y_0 being an arbitrary positive number.

Now the series $$\sum_1^\infty e^{-ny_0}$$

is convergent and its terms are independent of x and y.

Thus the series $v=a_1e^{-y}\sin x+a_2e^{-2y}\sin 2x+\dots$, ...(1)

regarded as a function of x, is uniformly convergent for any interval of x, when $y>0$; and, regarded as a function of y, it is uniformly convergent when $y\geqq y_0>0$.

The same is true of the series obtained by term by term differentiation of (1) with respect to x and y in these intervals.

Therefore
$$\frac{\partial^2 v}{\partial x^2} = -\sum_1^\infty n^2 a_n e^{-ny} \sin nx$$

and
$$\frac{\partial^2 v}{\partial y^2} = \sum_1^\infty n^2 a_n e^{-ny} \sin nx.$$

Hence
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Further, since (1) is uniformly convergent in the interval $0 \leqq x \leqq$ and the sum of the series vanishes when $x=0$ and $x=\pi$, the li of v as x approaches these values is zero, y being positive. Th the boundary conditions at the faces $x=0$ and $x=\pi$ are satisfied.

We have assumed that $f(x)$ is bounded and satisfies Dirichle Conditions in the interval $(0, \pi)$.

Therefore the Sine Series
$$a_1 \sin x + a_2 \sin 2x + \dots$$
converges, and its sum is $f(x)$ at every point between 0 and π which $f(x)$ is continuous, and $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at all other poin

It follows from *F.S.* § 73, I.* that, if v is defined by the series (
$$\underset{y\to 0}{\text{Lt}}\ v = f(x) \text{ at a point of continuity}$$
$$= \tfrac{1}{2}\{f(x+0)+f(x-0)\} \text{ at all other points.}$$

Thus
$$v = \sum_1^\infty a_n e^{-ny} \sin nx$$
is the solution of our problem.

This may be written
$$v = \frac{2}{\pi}\int_0^\pi f(x') \sum_1^\infty (e^{-ny} \sin nx \sin nx')\, dx',$$
since the series under the integral is uniformly convergent.

Ex. If the solid is bounded by the planes $x=0$, $x=a$, and $y=b$, wh are kept at zero temperature, and $y=0$, which is kept at temperature f show that
$$v = \frac{2}{a}\sum_1^\infty \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi}{a} b} \sin \frac{n\pi}{a} x \int_0^a f(x') \sin \frac{n\pi}{a} x'\, dx',$$
and discuss the other three cases in each of which three of the boundar are kept at zero temperature and the fourth at an arbitrary temperature

Cf. Byerly, *Fourier's Series and Spherical, Cylindrical and Ellipsoi Harmonics*, pp. 102-104.

* The argument of *F.S.* § 73, II. also applies to this question. Cf. above, §

43. Base at Temperature Unity.

In the solution of § 42, put $f(x)=1$, and we have

$$v=\frac{2}{\pi}\sum_1^\infty \frac{1-\cos n\pi}{n}e^{-ny}\sin nx.$$

Therefore

$$\frac{\pi}{4}v=e^{-y}\sin x+\tfrac{1}{3}e^{-3y}\sin 3x+\ldots$$

$$=\text{the imaginary part of } (e^{-y+ix}+\tfrac{1}{3}e^{-3y+3ix}+\ldots)$$

$$=\ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{2}\log\frac{1+e^{i(x+iy)}}{1-e^{i(x+iy)}}$$

$$=\ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{2}\log\left(\frac{1-e^{-2y}+2ie^{-y}\sin x}{1-2e^{-y}\cos x+e^{-2y}}\right)$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{\sinh y}\right).$$

Thus $$v=\frac{2}{\pi}\tan^{-1}\left(\frac{\sin x}{\sinh y}\right).$$

The conjugate function* to $\frac{2}{\pi}\tan^{-1}\left(\frac{\sin x}{\sinh y}\right)$ is

$$\frac{1}{4}\log\left(\frac{1+2e^{-y}\cos x+e^{-2y}}{1-2e^{-y}\cos x+e^{-2y}}\right)$$

$$\frac{1}{4}\log\left(\frac{\cosh y+\cos x}{\cosh y-\cos x}\right).$$

It follows that the lines of flow are given by

$$\frac{\cosh y+\cos x}{\cosh y-\cos x}=\text{constant},$$

these being orthogonal to the isothermals

$$\frac{\sin x}{\sinh y}=\text{constant}.\dagger$$

44. The Use of Conjugate Functions in Problems of Steady Temperature.

Let ξ, η be real functions of x and y such that

$$\xi+i\eta=f(x+iy)=f(z).$$

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* For the definition and properties of conjugate functions see § 44.

† Cf. Fourier, *loc. cit.*, § 205.

Then ξ, η are called conjugate functions of x and y. Also we have

$$\frac{\partial \xi}{\partial x}+i\frac{\partial \eta}{\partial x}=f'(z),$$

$$\frac{\partial \xi}{\partial y}+i\frac{\partial \eta}{\partial y}=if'(z).$$

Therefore
$$\frac{\partial \xi}{\partial x}=-\frac{\partial \eta}{\partial y}, \quad \text{.........(1)}$$

$$\frac{\partial \eta}{\partial x}=\frac{\partial \xi}{\partial y}. \quad \text{.........(2)}$$

It follows that the curves ξ=constant and η=constant are orthogonal.

Again, since
$$\frac{\partial^2 \xi}{\partial x^2}=-\frac{\partial^2 \eta}{\partial x\,\partial y},$$

and
$$\frac{\partial^2 \xi}{\partial y^2}=\frac{\partial^2 \eta}{\partial x\,\partial y},$$

it follows that
$$\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}=0, \quad \text{.........(3)}$$

and similarly
$$\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}=0. \quad \text{.........(4)}$$

Further, if v is a function of x and y such that

$$\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}=0, \quad \text{.........(5)}$$

we can show that
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

For we have
$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x}$$

and

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2+2\frac{\partial^2 v}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial^2 v}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2+\frac{\partial v}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial v}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}.$$

Similarly

$$\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2+2\frac{\partial^2 v}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}+\frac{\partial^2 v}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2+\frac{\partial v}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial v}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}.$$

Adding these two results, and using (1), (2), (3), (4), and (5), we see that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

Thus, if we can obtain a solution of the equation

$$\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}=0,$$

satisfying certain boundary conditions at the curves

$$\xi=\xi_1,\quad \xi=\xi_2,$$
$$\eta=\eta_1,\quad \eta=\eta_2,$$

this solution in the $\xi\eta$ plane may be transferred to the xy plane, the boundaries being the curves in the xy plane which correspond by the transformation

$$\xi+i\eta=f(x+iy)$$

to the curves $\xi=\xi_1$, etc., while the temperatures at these boundaries correspond to the temperatures at the boundaries in the $\xi\eta$ plane.

Suppose that we have the case of the rectangle in the $\xi\eta$ plane given by

$$\xi=\xi_1,\quad \xi=\xi_2,$$
$$\eta=\eta_1,\quad \eta=\eta_2,$$

and that

$$\begin{aligned} v&=f_1(\eta) \text{ at } \xi=\xi_1, & (\eta_1<\eta<\eta_2)\\ v&=f_2(\eta) \text{ at } \xi=\xi_2, & (\eta_1<\eta<\eta_2)\\ v&=F_1(\xi) \text{ at } \eta=\eta_1, & (\xi_1<\xi<\xi_2)\\ v&=F_2(\xi) \text{ at } \eta=\eta_2. & (\xi_1<\xi<\xi_2)\end{aligned}$$

The solution of this problem is obtained by breaking it up into four cases, in each of which three of the boundaries are kept at zero temperature. In this way we find

$$v=\sum_1^\infty \frac{a_n \sinh\dfrac{n\pi(\xi_2-\xi)}{(\eta_2-\eta_1)}+a_n'\sinh\dfrac{n\pi(\xi-\xi_1)}{(\eta_2-\eta_1)}}{\sinh\dfrac{n\pi(\xi_2-\xi_1)}{(\eta_2-\eta_1)}}\sin\frac{n\pi(\eta-\eta_1)}{(\eta_2-\eta_1)}$$

$$+\sum_1^\infty \frac{b_n \sinh\dfrac{n\pi(\eta_2-\eta)}{(\xi_2-\xi_1)}+b_n'\sinh\dfrac{n\pi(\eta-\eta_1)}{(\xi_2-\xi_1)}}{\sinh\dfrac{n\pi(\eta_2-\eta_1)}{(\xi_2-\xi_1)}}\sin\frac{n\pi(\xi-\xi_1)}{(\xi_2-\xi_1)},$$

where a_n, a_n', b_n, and b_n' are the coefficients in the Sine Series,

$$f_1(\eta)=\sum_1^\infty a_n \sin\frac{n\pi(\eta-\eta_1)}{(\eta_2-\eta_1)},$$

$$f_2(\eta)=\sum_1^\infty a'_n \sin\frac{n\pi(\eta-\eta_1)}{(\eta_2-\eta_1)},$$

$$F_1(\xi)=\sum_1^\infty b_n \sin\frac{n\pi(\xi-\xi_1)}{(\xi_2-\xi_1)},$$

$$F_2(\xi)=\sum_1^\infty b_n' \sin\frac{n\pi(\xi-\xi_1)}{(\xi_2-\xi_1)}.$$

Substituting for ξ, η from the relation

$$\xi+i\eta=f(x+iy),$$

we have the temperature in the region bounded by the curves which correspond to $\xi=\xi_1$, etc., these curves being kept at the temperature corresponding to $f_1(\eta)$, etc.

45. Applications of this Method.*

I. *The Sector of a Circle.*

Consider the transformation

$$\xi+i\eta=-\frac{i\pi}{a}\log\frac{x+iy}{a}.$$

In this case
$$\xi=\frac{\pi}{a}\theta,\quad \eta=\frac{\pi}{a}\log\left(\frac{a}{r}\right),$$

and the sector of radius a and angle a corresponds to the region

$$0<\eta,\quad 0<\xi<\pi,$$

in the $\xi\eta$ plane.

Thus the equations

$$\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}=0,\quad \left.\begin{matrix}(0<\xi<\pi)\\(0<\eta)\end{matrix}\right\}$$

$$v=0,\ \text{when } \xi=0 \text{ and } \xi=\pi,$$

and
$$v=1,\ \text{when } \eta=0,$$

lead to
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \text{ over the sector,}$$

$$v=0,\ \text{when } \theta=0 \text{ and } \theta=a,$$

and
$$v=1,\ \text{when } r=a.$$

These equations in (ξ, η) we have already discussed in § 43 when dealing with the Infinite Rectangular Solid, and their solution is

$$v=\frac{2}{\pi}\tan^{-1}\left(\frac{\sin\xi}{\sinh\eta}\right).$$

Therefore the temperature in the sector is given by

$$v=\frac{2}{\pi}\tan^{-1}\left\{\frac{\sin\dfrac{\pi}{a}\theta}{\sinh\left(\dfrac{\pi}{a}\log\dfrac{a}{r}\right)}\right\}.$$

---

* Cf. Mathieu, *Cours de Physique Mathématique*, Ch. III.

If the boundary $r=a$ had been kept at $v=f(\theta)$, the problem would have reduced to the solution of the equations

$$\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}=0,$$

$$v=0, \text{ when } \xi=0 \text{ and } \xi=\pi,$$

and
$$v=f\left(\frac{a}{\pi}\xi\right), \text{ when } \eta=0,$$

and this has been discussed in § 42.

II. *The Circle.*

Consider the transformation

$$\xi+i\eta=-i\log\frac{x+iy}{a}.$$

Then
$$\xi=\theta, \quad \eta=\log\frac{a}{r},$$

and the circle $r=a$ corresponds to the region

$$0<\eta, \quad 0<\xi<2\pi$$

of the $\xi\eta$ plane.

Thus the equations

$$\left.\begin{aligned}\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}&=0 \text{ over this region,}\\ v=f(\xi), &\text{ when } \eta=0,\end{aligned}\right\} \qquad \text{.................(1)}$$

lead to
$$\left.\begin{aligned}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}&=0 \text{ in the circle,}\\ v=f(\theta), &\text{ when } r=a.\end{aligned}\right\} \qquad \text{.................(2)}$$

The solution of (1) is given by

$$v=\sum_0^\infty e^{-n\eta}(a_n\cos n\xi+b_n\sin n\xi),$$

where
$$a_0=\frac{1}{2\pi}\int_0^{2\pi} f(\xi')\,d\xi',$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(\xi')\cos n\xi'\,d\xi',$$

and
$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(\xi')\sin n\xi'\,d\xi'.$$

Thus we have

$$v=\frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sum_1^\infty e^{-n\eta}\cos n(\xi-\xi')\right)f(\xi')\,d\xi'$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-e^{-2\eta}}{1-2e^{-\eta}\cos(\xi-\xi')+e^{-2\eta}}f(\xi')\,d\xi'.$$

Therefore the temperature in the circle is given by

$$v=\frac{1}{2\pi}\int_0^{2\pi}f(\theta')\frac{a^2-r^2}{a^2-2ar\cos(\theta-\theta')+r^2}\,d\theta'.$$

III. *Two Concentric Circles.*

This may be obtained from the same transformation. T solution can readily be found in the form

$$v=\sum_0^\infty\frac{\sinh n(\eta_2-\eta)}{\sinh n(\eta_2-\eta_1)}(a_n\cos n\xi+b_n\sin n\xi)$$

$$+\sum_0^\infty\frac{\sinh n(\eta-\eta_1)}{\sinh n(\eta_2-\eta_1)}(a_n'\cos n\xi+b_n'\sin n\xi),$$

where $$f_1(\xi)=\sum_0^\infty(a_n\cos n\xi+b_n\sin n\xi),$$

$$f_2(\xi)=\sum_0^\infty(a_n'\cos n\xi+b_n'\sin n\xi),$$

are the Fourier's Series for $f_1(\xi)$ and $f_2(\xi)$ in the interval 0 to 2

IV. *Two Intersecting or Non-Intersecting Circles.*

Consider the transformation

$$\xi+i\eta=\log\frac{x+1-iy}{x-1-iy}.$$

Then $$\xi=\log\frac{r_2}{r_1},\quad \eta=\theta_1-\theta_2,$$

where r_1 and r_2 are the distances from the points $A(1, 0)$, $B(-1$ to the point $P(x, y)$, and θ_1, θ_2 are the angles AP and BP m with the positive direction of the axis of x.

Thus ξ=constant represents the system of coaxal circles w A, B as limiting points, and η=constant represents the systen circles passing through A, B, these two sets of curves, as in cases of conjugate functions, being orthogonal. With this nota the xy plane is given by $-\pi<\eta<\pi$ and $-\infty<\xi<\infty$; the lo side of BA is $\eta=-\pi$; the lines Ax and Bx' are $\eta=0$; and

upper side of BA is $\eta=\pi$. Also A is the point $\xi=+\infty$, and B is the point $\xi=-\infty$.

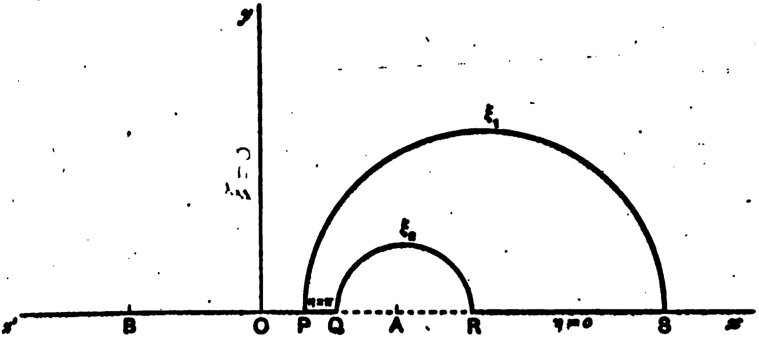

Fig. 6.

We proceed to apply this transformation to several cases in which the region in the xy plane is bounded by arcs of these circles.

(i) Consider the region bounded by

$$\xi=\xi_1 \text{ and } \xi=\xi_2, \quad (0<\eta<\pi)$$
$$\eta=0 \text{ and } \eta=\pi, \quad (\xi_1<\xi<\xi_2)$$

as in Fig. 6.

Let $\quad v=0 \quad$ over $\xi=\xi_2$, $\eta=0$, and $\eta=\pi$,

and $\quad v=f(\eta) \quad$ over $\xi=\xi_1$.

Then we have $\quad v=\sum_1^\infty a_n \dfrac{\sinh n(\xi_2-\xi)}{\sinh n(\xi_2-\xi_1)} \sin n\eta,$

where $\quad a_n=\dfrac{2}{\pi}\displaystyle\int_0^\pi f(\eta')\sin n\eta'\,d\eta'.$

It is easy to extend this solution to the case where

$$v=f_1(\eta) \quad \text{over } \xi=\xi_1, \quad (0<\eta<\pi)$$
$$v=f_2(\eta) \quad \text{over } \xi=\xi_2, \quad (0<\eta<\pi)$$
$$v=F_1(\xi) \quad \text{over } \eta=0, \quad (\xi_1<\xi<\xi_2)$$
$$v=F_2(\xi) \quad \text{over } \eta=\pi. \quad (\xi_1<\xi<\xi_2)$$

(ii) Consider the region bounded by the two complete circles $\xi=\xi_1$ and $\xi=\xi_2$ surrounding the limiting point A.

Let $v=f_1(\eta)$ over $\xi=\xi_1$, and $v=0$ over $\xi=\xi_2$. Then the solution is obviously

$$v=\sum_0^\infty \frac{\sinh n(\xi_2-\xi)}{\sinh n(\xi_2-\xi_1)}(a_n\cos n\eta+b_n\sin n\eta),$$

where
$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\eta')\,d\eta')$$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\eta')\cos n\eta'\,d\eta',$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\eta')\sin n\eta'\,d\eta'.$$

Similarly, when $v=f_2(\eta)$ over $\xi=\xi_2$ and $v=0$ over $\xi=\xi_1$, we have
$$v=\sum_0^{\infty}\frac{\sinh n(\xi-\xi_1)}{\sinh n(\xi_2-\xi_1)}(a_n'\cos n\eta+b_n'\sin n\eta'),$$
where a_n' and b_n' are the coefficients in the Fourier's Series for $f_2(\eta)$ in the interval $-\pi$ to π.

Adding these two results we have the solution for the case of the circles ξ_1 and ξ_2 at temperatures $f_1(\eta)$ and $f_2(\eta)$.

It is clear that if $f_1(\eta)$ and $f_2(\eta)$ are constant and equal to v_1 and v_2 respectively, we have only to solve the equations
$$\left.\begin{aligned}&\frac{\partial^2 v}{\partial \xi^2}=0,\\ &v=v_1,\ \text{when}\ \xi=\xi_1,\\ &v=v_2,\ \text{when}\ \xi=\xi_2.\end{aligned}\right\}$$

The solution is $\quad v=v_1\left(\dfrac{\xi_2-\xi}{\xi_2-\xi_1}\right)+v_2\left(\dfrac{\xi-\xi_1}{\xi_2-\xi_1}\right).$

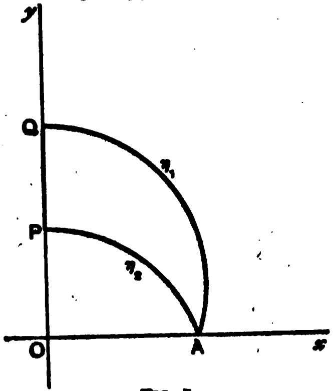

FIG. 7.

(iii) Consider the region bounded by
$$\eta=\eta_1,\quad (0<\xi<\infty)$$
$$\eta=\eta_2,\quad (0<\xi<\infty)$$
$$\xi=0,\quad (\eta_1<\eta<\eta_2)$$
as in Fig. 7.

(α) Let
$$v=f_1(\eta) \text{ over } \xi=0, \quad (\eta_1<\eta<\eta_2)$$
$$v=0 \quad \text{over } \eta=\eta_1, \quad (0<\xi<\infty).$$
$$v=0 \quad \text{over } \eta=\eta_2. \quad (0<\xi<\infty)$$

Then the solution is clearly
$$v=\sum_1^\infty a_n e^{-n\xi}\sin\frac{(\eta-\eta_1)}{(\eta_2-\eta_1)}n\pi,$$
where $a_n=\dfrac{2}{\eta_2-\eta_1}\displaystyle\int_{\eta_1}^{\eta_2} f_1(\eta')\sin\frac{(\eta'-\eta_1)}{(\eta_2-\eta_1)}n\pi\,d\eta'$. (Cf. *F.S.*, § 98, (7).)

(β) Again, let
$$v=f_2(\xi) \text{ over } \eta=\eta_1, \quad (0<\xi<\infty)$$
$$v=0 \quad \text{over } \eta=\eta_2, \quad (0<\xi<\infty)$$
$$v=0 \quad \text{over } \xi=0. \quad (0<\xi<\infty)$$

Suppose in the first place that the point $A\,(\xi=\infty)$ is excluded by a circle $\xi=a$, where a is a very large positive quantity, and that the part of this circle included in the region is kept at zero temperature.

Then we would have
$$v=\sum_1^\infty a_n'\frac{\sinh\dfrac{n\pi}{a}(\eta_2-\eta)}{\sinh\dfrac{n\pi}{a}(\eta_2-\eta_1)}\sin\frac{n\pi}{a}\xi,$$
where
$$a_n'=\frac{2}{a}\int_0^a f_2(\xi')\sin\frac{n\pi}{a}\xi'\,d\xi'.$$

To obtain the solution of our problem, we must let a tend to ∞. Proceeding as in *F.S.* § 118, in the case of Fourier's Integral, put
$$\frac{n\pi}{a}=\lambda \quad \text{and} \quad \frac{\pi}{a}=\Delta\lambda.$$

Then $$v=\frac{2}{\pi}\int_0^\infty d\lambda\frac{\sinh\lambda(\eta_2-\eta)}{\sinh\lambda(\eta_2-\eta_1)}\sin\lambda\xi\int_0^\infty f_2(\xi')\sin\lambda\xi'\,d\xi'.$$

This result might have been deduced from Fourier's Sine Integral for $f_2(\xi)$, namely,
$$\frac{2}{\pi}\int_0^\infty d\lambda\sin\lambda\xi\int_0^\infty f_2(\xi')\sin\lambda\xi'\,d\xi'.$$

(γ) In the same way, if
$$v=f_3(\xi), \text{ when } \eta=\eta_2, \quad (0<\xi<\infty)$$
$$v=0, \quad \text{when } \eta=\eta_1, \quad (0<\xi<\infty)$$
$$v=0, \quad \text{when } \xi=0, \quad (\eta_1<\eta<\eta_2)$$

we have $$v=\frac{2}{\pi}\int_0^\infty d\lambda\,\frac{\sinh\lambda(\eta-\eta_1)}{\sinh\lambda(\eta_2-\eta_1)}\sin\lambda\xi\int_0^\infty f_3(\xi')\sin\lambda\xi'\,d\xi'.$$

(δ) By adding (α), (β), and (γ), we find the solution when the three surfaces are kept at $v=f_1(\eta)$, $v=f_2(\xi)$, and $v=f_3(\xi)$.

(iv) Consider the region bounded by

$$\eta=\eta_1, \quad (-\infty<\xi<\infty)$$
$$\eta=\eta_2. \quad (-\infty<\xi<\infty)$$

Let $$v=f_1(\xi) \text{ over } \eta=\eta_1$$
and $$v=f_2(\xi) \text{ over } \eta=\eta_2.$$

Then we find, as above,

$$v=\frac{1}{\pi}\int_0^\infty d\lambda\,\frac{\sinh\lambda(\eta_2-\eta)}{\sinh\lambda(\eta_2-\eta_1)}\int_{-\infty}^\infty f_1(\xi')\cos\lambda(\xi'-\xi)\,d\xi'$$
$$+\frac{1}{\pi}\int_0^\infty d\lambda\,\frac{\sinh\lambda(\eta-\eta_1)}{\sinh\lambda(\eta_2-\eta_1)}\int_{-\infty}^\infty f_2(\xi')\cos\lambda(\xi'-\xi)\,d\xi.$$

V. *Confocal Ellipses or Hyperbolas.*

Consider the transformation

$$\xi+i\eta=\cosh^{-1}\frac{x+iy}{c},$$
or $$x+iy=c\cosh(\xi+i\eta).$$

Then $$x=c\cosh\xi\cos\eta, \quad y=c\sinh\xi\sin\eta,$$
and $$\frac{x^2}{\cosh^2\xi}+\frac{y^2}{\sinh^2\xi}=c^2,$$
$$\frac{x^2}{\cos^2\eta}-\frac{y^2}{\sin^2\eta}=c^2.$$

Thus the curves $\xi=$constant, and $\eta=$constant, are a set of confocal ellipses and hyperbolas, and the xy plane is given by $-\pi<\eta<\pi$ and $0<\xi<\infty$, the lower part of the xy plane having negative values of η and the upper part positive values.

(i) *Two Confocal Ellipses.*

Consider the region bounded by $\xi=\xi_1$ and $\xi=\xi_2$.

Let $$v=f_1(\eta) \text{ over } \xi=\xi_1,$$
$$v=f_2(\eta) \text{ over } \xi=\xi_2.$$

Then, as above,

$$v=\sum_0^\infty\frac{\sinh n(\xi_2-\xi)}{\sinh n(\xi_2-\xi_1)}(a_n\cos n\eta+b_n\sin n\eta)$$
$$+\sum_0^\infty\frac{\sinh n(\xi-\xi_1)}{\sinh n(\xi_2-\xi_1)}(a_n'\cos n\eta+b_n'\sin n\eta),$$

where a_n, b_n, a_n', and b_n' are the coefficients in the Fourier's Series for $f_1(\eta)$ and $f_2(\eta)$ in the interval $-\pi$ to π.

(ii) *Two Semi-Ellipses and the part of the major axis between them.*

In this case the region is bounded by

$$\xi=\xi_1 \text{ and } \xi=\xi_2, \quad (0<\eta<\pi)$$
$$\eta=0 \text{ and } \eta=\pi. \quad (\xi_1<\xi<\xi_2)$$

Let $\quad v=f_1(\eta)$ over $\xi=\xi_1$,

$\quad v=f_2(\eta)$ over $\xi=\xi_2$,

and $\quad v=0$ over $\eta=0$ and $\eta=\pi$.

It is clear that the solution is

$$v=\sum_1^\infty a_n \sin n\eta \frac{\sinh n(\xi_2-\xi)}{\sinh n(\xi_2-\xi_1)} + \sum_1^\infty a_n' \sin n\eta \frac{\sinh n(\xi-\xi_1)}{\sinh n(\xi_2-\xi_1)},$$

where a_n and a_n' are the coefficients in the Sine Series for $f_1(\eta)$ and $f_2(\eta)$.

(iii) *Semi-Ellipse.*

In this case the region is bounded by

$$\xi=0 \text{ and } \xi=\xi_1, \quad (0<\eta<\pi)$$
$$\eta=0 \text{ and } \eta=\pi. \quad (0<\xi<\xi_1).$$

Let $v=f(\eta)$ over $\xi=\xi_1$, and let the major axis be at zero temperature. Then

$$v=\sum_1^\infty a_n \sin n\eta \frac{\sinh n\xi}{\sinh n\xi_1},$$

where a_n is the coefficient in the Sine Series for $f(\eta)$.

(iv) *Complete Ellipse.*

In this case we have to satisfy

$$\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}=0, \quad \left.\begin{matrix}(0<\xi<\xi_1)\\(-\pi<\eta<\pi)\end{matrix}\right\}$$
$$v=f(\eta), \text{ when } \xi=\xi_1. \quad (-\pi<\eta<\pi)$$

Also there must be no discontinuity in the temperature or the flow of heat as we cross the major axis or pass along it.

All these conditions are satisfied by the expression

$$v=\sum_{n=0}^\infty \left(a_n \frac{\cosh n\xi}{\cosh n\xi_1} \cos n\eta + b_n \frac{\sinh n\xi}{\sinh n\xi_1} \sin n\eta\right),$$

where a_n and b_n are the coefficients in the Fourier's Series for $f(\eta)$ in the interval $-\pi$ to π.

(v) *Quadrilateral bounded by the Arcs of two Confocal Ellipses and Hyperbolas.*

This reduces to the rectangle in the $\xi\eta$ plane, and the solution follows.

46. Sources and Sinks in Steady Temperature.

In the cases of steady motion considered in the preceding articles, the supply of heat which maintains the steady temperature is applied at the boundaries of the solid. On the analogy of the flow of electric currents along thin conducting plates, when the current is conveyed to the plate by one electrode, and withdrawn by another, we may imagine the steady flow of heat in two dimensions—or the flow of heat in thin plates—to be produced by the introduction of a quantity of heat at one or more points and its withdrawal at others. These points may be called Sources and Sinks of Heat.

In this case, if we describe a small circle of radius r round a point at which heat is steadily flowing into the plate, in the limit the flow of heat out through this circle must be equal to the flow in at the source. Hence the solution of our equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

must take the form
$$-\frac{m}{2K\pi}\log r+u,$$

where u is a solution which remains finite as the point approaches the source, and m is the quantity of heat introduced there per unit time.

Similarly at a sink the part of v which tends to infinity must be equal to $\dfrac{m}{2K\pi}\log r$.

Consider the part of the xy plane for which $y>0$, and let the boundary $y=0$ be kept at temperature zero, while there is a source of strength m at $(0, y_0)$.

Then we have
$$v=\frac{m}{4K\pi}\log\frac{x^2+(y+y_0)^2}{x^2+(y-y_0)^2},$$

since this expression satisfies all the conditions of the problem.

It will be seen that this solution is obtained by putting a sink

at $(0, -y_0)$, which will balance the source at $(0, y_0)$. Indeed the use of Sources and Sinks in two-dimensional problems of Steady Temperature is exactly analogous to the use of Images in Electrostatics and Hydrodynamics, and the reader is referred to the discussions in the books on these subjects for further illustrations.

It is clear that the method of transformation by conjugate functions is also applicable in this connection. Since if

$$\phi+i\psi=\frac{m}{2K\pi}\log\left(\frac{Z-Z_0'}{Z-Z_0}\right),$$

where
$$Z=X+iY=f(x+iy),$$
$$Z_0=X_0+iY_0=f(x_0+iy_0),$$
and
$$Z_0'=X_0-iY_0=f(x_0-iy_0),$$

ϕ, looked upon as a function of (x, y), satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0.$$

Also ϕ vanishes at the boundary which corresponds to $Y=0$ in the XY plane, and is infinite as $-\frac{m}{2K\pi}\log r$ at the point (x_0, y_0), which corresponds to the point (X_0, Y_0) in the XY plane.

47. Variable Temperature.

In this case the equation of conduction is

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),$$

and we obtain the solution as in § 16 in the form

$$v=\frac{1}{4\pi\kappa t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x', y')e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}\,dx'\,dy',$$

the initial temperature being

$$v=f(x, y).$$

If the solid is bounded by $y=0$ kept at temperature zero and the initial temperature is $v=f(x, y)$, when $-\infty<x<\infty$ and $0<y$, we have

$$v=\frac{1}{4\pi\kappa t}\int_{-\infty}^{\infty}\int_{0}^{\infty}f(x', y')\left\{e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}-e^{-\frac{(x-x')^2+(y+y')^2}{4\kappa t}}\right\}dx'\,dy'.$$

To obtain this solution we may suppose the solid continued beyond the plane $y=0$ and that there is a symmetrical initial

distribution of temperature in the added region $y < 0$ which w
cause the plane $y=0$ always to keep zero temperature: that
we take the initial temperature at $(x', -y')$, $(y' > 0)$, to be equal
absolute value but opposite in sign, to that at (x', y').

The question of a variable temperature at the boundary $y=$
or of radiation across it, will be referred to in §§ 74, 86 and 87.

CHAPTER VI

THE FLOW OF HEAT IN A RECTANGULAR PARALLELEPIPED

48. Introductory.

Several of the methods of finding the conductivity of solids which we have discussed in the previous chapters cannot be applied to poor conductors, since the amount of heat lost at the surface of the bar by radiation becomes large compared with that conducted along the bar, and the emissivity is such an uncertain quantity that it is best to have it, when possible, reduced to the rank of a small correction. With poor conductors this would be impossible in the bar methods. But for the cube, the sphere, and the cylinder the mathematical problem can be solved and its solution applied to the evaluation of the thermal constants. In this chapter we shall discuss the case of the rectangular parallelepiped. For steady temperature the solution is given by a rather complicated series, without much practical value, but for different problems of variable temperature we obtain results immediately applicable to experimental investigation.

49. Steady Temperature.

Consider the solid bounded by the planes $x=0$, $x=a$; $y=0$, $y=b$; $z=0$, $z=c$. The equation for the temperature is

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0.$$

We take the following surface conditions :—

$$v=v_1, \text{ when } x=0,$$

$$v=v_2, \text{ when } x=a,$$

and the other faces at zero.

It is clear that the expression

$$\frac{v_1 \sinh l(a-x)+v_2 \sinh lx}{\sinh la} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}$$

satisfies all the conditions provided that

$$l^2=\left(\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\pi^2.$$

Therefore the solution of the problem is given by

$$v=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\frac{v_1 \sinh l(a-x)+v_2 \sinh lx}{\sinh la} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c},$$

provided that $\displaystyle\sum_1^{\infty}\sum_1^{\infty} A_{m,n} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}=1.$*

But
$$\sum_{m=1}^{\infty} a_m \sin \frac{m\pi y}{b}=1,$$

when
$$a_m=\frac{2}{m\pi}(1-\cos m\pi).$$

Therefore

$$v=\frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{v_1 \sinh l(a-x)+v_2 \sinh lx}{\sinh la}\frac{\sin \frac{(2p+1)\pi y}{b}}{(2p+1)}\frac{\sin \frac{(2q+1)\pi z}{c}}{(2q+1)},$$

where
$$\frac{l^2}{\pi^2}=\frac{(2p+1)^2}{b^2}+\frac{(2q+1)^2}{c^2}.$$

50. Steady Temperature (*continued*).

When radiation takes place at the faces $y=0$, $y=b$, $z=0$, $z=c$, and the faces $x=0$, $x=a$ are kept at temperatures v_1, v_2, as before, we can obtain a similar solution of the problem.

In this case the surface conditions are

$$v=v_1 \text{ when } x=0, \text{ and } v=v_2 \text{ when } x=a;$$

$$-\frac{\partial v}{\partial y}+hv=0, \text{ when } y=0; \quad \frac{\partial v}{\partial y}+hv=0, \text{ when } y=b;$$

$$-\frac{\partial v}{\partial z}+hv=0, \text{ when } z=0; \quad \frac{\partial v}{\partial z}+hv=0, \text{ when } z=c.$$

---

* For a rigorous treatment of this question it would be necessary to justify the term by term differentiation of the double series for v. The same remark applies to the other problems discussed in this chapter.

Cf. §§ 12, 30 above, and Moore: "On Convergence Factors in Double Series and the Double Fourier's Series," *Trans. Amer. Math. Soc.*, 14, p. 99, 1913. Also the same author's paper in *Bull. Amer. Math. Soc.*, 25, p. 274, 1919.

The expression

$$\frac{v_1 \sinh l(a-x)+v_2 \sinh lx}{\sinh la}\left(\cos my+\frac{h}{m}\sin my\right)\left(\cos nz+\frac{h}{n}\sin nz\right)$$

satisfies the equation of conduction provided that

$$l^2=m^2+n^2.$$

Also it satisfies the surface conditions at $x=0$ and $x=a$ for all values of l, and those at the other faces, if m, n are the positive roots of

$$\tan mb=\frac{2mh}{m^2-h^2} \quad \text{and} \quad \tan nc=\frac{2nh}{n^2-h^2}$$

(cf. § 36).

Now we have seen in § 36, that, with certain assumptions as to the possibility of the expansion of an arbitrary function in a series whose terms are of the type

$$Y_r=\cos m_r y+\frac{h}{m_r}\sin m_r y,$$

the coefficients in the expansion

$$f(y)=A_1Y_1+A_2Y_2+\ldots$$

are given by
$$A_r=\frac{2m_r^2}{(m_r^2+h^2)b+2h}\int_0^b f(y)\,Y_r\,dy.$$

But
$$\int_0^b Y_r\,dy=\int_0^b\left(\cos m_r y+\frac{h}{m_r}\sin m_r y\right)dy$$

$$=\frac{1}{m_r}\left(\sin m_r b+\frac{h}{m_r}(1-\cos m_r b)\right).$$

Also,
$$\tan\tfrac{1}{2}m_{2r}b=-\frac{m_{2r}}{h},$$

$$\tan\tfrac{1}{2}m_{2r+1}b=\frac{h}{m_{2r+1}}$$

(cf. § 37).

Hence
$$\int_0^b Y_r\,dy=0, \text{ when } r \text{ is even};$$

$$=\frac{2h}{m_r^2}, \text{ when } r \text{ is odd};$$

and
$$1=\frac{4h}{b(m_1^2+h^2)+2h}\,Y_1+\frac{4h}{b(m_3^2+h^2)+2h}\,Y_3+\ldots.$$

Similar results follow for the corresponding expression Z_r.

Therefore the solution of our problem is

$$v=16h^2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{v_1\sinh l(a-x)+v_2\sinh lx}{\sinh la}$$
$$\times\frac{Y_{2p+1}}{b(m_{2p+1}^2+h^2)+2h}\,\frac{Z_{2q+1}}{c(n_{2q+1}^2+h^2)+2h},$$

where

$$l^2=m_{2p+1}^2+n_{2q+1}^2.$$

It is clear that these solutions do not lend themselves to numerical calculation, and that they are not suitable for the evaluation of the thermal constants.

51. Variable Temperature. No Radiation at the Surface.

Let the solid be the rectangular parallelepiped of the last two sections, the initial temperature being an arbitrary function of x, y and z, and all the faces being kept at zero temperature. Then we have to satisfy the equations,

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v \text{ in the solid}; \quad\text{.........(1)}$$

$$v=0, \text{ when } \left.\begin{matrix} x=0, & y=0, & z=0\\ x=a, & y=b, & z=c\end{matrix}\right\}; \quad\text{.........(2)}$$

and

$$v=f(x, y, z), \text{ when } t=0. \quad\text{.........(3)}$$

The expression

$$e^{-\kappa\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)t}\sin\frac{l\pi}{a}x\sin\frac{m\pi}{b}y\sin\frac{n\pi}{c}z$$

satisfies (1) and (2).

Extending Fourier's Sine Series to the case of $f(x, y, z)$, we would have

$$f(x, y, z)=\frac{8}{abc}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{l\pi}{a}x\sin\frac{m\pi}{b}y\sin\frac{n\pi}{c}z$$
$$\times\left\{\int_0^a\int_0^b\int_0^c f(x', y', z')\sin\frac{l\pi}{a}x'\sin\frac{m\pi}{b}y'\sin\frac{n\pi}{c}z'\,dx'\,dy'\,dz'\right\}$$
$$=\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{l,m,n}\sin\frac{l\pi}{a}x\sin\frac{m\pi}{b}y\sin\frac{n\pi}{c}z, \text{ say.}$$

Hence the solution of our problem will be given by

$$v=\sum_1^{\infty}\sum_1^{\infty}\sum_1^{\infty}A_{l,m,n}\sin\frac{l\pi}{a}x\sin\frac{m\pi}{b}y\sin\frac{n\pi}{c}z\,e^{-\kappa\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)t},$$

$A_{l,m,n}$ being the definite integral

$$\frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x', y', z')\sin\frac{l\pi}{a}x'\sin\frac{m\pi}{b}y'\sin\frac{n\pi}{c}z'\,dx'\,dy'\,dz'.$$

When $f(x, y, z)$ is constant and equal to v_0,

$$A_{l,m,n}=\frac{8v_0}{abc}\int_0^a\int_0^b\int_0^c \sin\frac{l\pi}{a}x\sin\frac{m\pi}{b}y\sin\frac{n\pi}{c}z\,dx\,dy\,dz$$

$$=\frac{8v_0}{lmn\pi^3}(1-\cos l\pi)(1-\cos m\pi)(1-\cos n\pi).$$

Therefore, in this case,

$$\frac{v}{v_0}=\left(\frac{4}{\pi}\right)^3\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\left\{\sin(2p+1)\frac{\pi x}{a}\sin(2q+1)\frac{\pi y}{b}\sin(2r+1)\frac{\pi z}{c}\right.$$

$$\left.\times\frac{e^{-\kappa\pi^2\left(\frac{(2p+1)^2}{a^2}+\frac{(2q+1)^2}{b^2}+\frac{(2r+1)^2}{c^2}\right)t}}{(2p+1)(2q+1)(2r+1)}\right\}$$

$$=\left(\frac{4}{\pi}\right)^3\left\{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}e^{-\kappa\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)t}\right.$$

$$+\frac{1}{3}\sin\frac{3\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}e^{-\kappa\pi^2\left(\frac{9}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)t}$$

$$+\frac{1}{3}\sin\frac{\pi x}{a}\sin\frac{3\pi y}{b}\sin\frac{\pi z}{c}e^{-\kappa\pi^2\left(\frac{1}{a^2}+\frac{9}{b^2}+\frac{1}{c^2}\right)t}$$

$$+\frac{1}{3}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{3\pi z}{b}e^{-\kappa\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{9}{c^2}\right)t}$$

$$\left.+\ldots\ldots\right\}.$$

Thus, for large values of t, the series is rapidly converging, and to a close approximation

$$\frac{v}{v_0}=\left(\frac{4}{\pi}\right)^3\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}e^{-\kappa\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)t}.$$

Also, we note that at the point

$$(\tfrac{1}{3}a,\ \tfrac{1}{3}b,\ \tfrac{1}{3}c),$$

the 2nd, 3rd, and 4th terms disappear, so that

$$\frac{v}{v_0}=\left(\frac{4}{\pi}\right)^3\left(\frac{\sqrt{3}}{2}\right)^3e^{-\kappa\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)t}$$

$$=\frac{24\sqrt{3}}{\pi^3}e^{-\kappa\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)t}$$

is a good approximation to the value of v at this point for large values of t.

In applying this result it is usual to take a cube, so that a, b and c are equal, and

$$v=\frac{24\sqrt{3}}{\pi^3}v_0e^{-3\kappa\frac{\pi^2}{a^2}t}.$$

That sufficient time has passed to allow this to represent the state of the temperature may be tested by seeing if the temperature readings at this point follow the exponential curve. It will be noticed that the 5th, 6th, and 7th terms are

$$-\frac{24\sqrt{3}}{5\pi^3}v_0 e^{-27\kappa\frac{\pi^2}{a^2}t},$$

and that these should be negligible.

If v_1 and v_2 are the readings of the temperature at t_1 and t_2, the equation

$$\frac{v_1}{v_2}=e^{3\kappa\frac{\pi^2}{a^2}(t_2-t_1)}$$

gives the value of κ.

52. Variable Temperature (*continued*). Radiation at the Surface.*

When there is radiation at the faces of the parallelepiped into a medium at zero temperature, we have to satisfy the equations,

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v \text{ in the solid,} \quad\text{(1)}$$

$$-\frac{\partial v}{\partial x}+hv=0, \text{ when } x=0, \quad \frac{\partial v}{\partial x}+hv=0, \text{ when } x=a; \quad\text{(2)}$$

$$-\frac{\partial v}{\partial y}+hv=0, \text{ when } y=0, \quad \frac{\partial v}{\partial y}+hv=0, \text{ when } y=b; \quad\text{(3)}$$

$$-\frac{\partial v}{\partial z}+hv=0, \text{ when } z=0, \quad \frac{\partial v}{\partial z}+hv=0, \text{ when } z=c; \quad\text{(4)}$$

and

$$v=f(x, y, z), \text{ when } t=0. \quad\text{(5)}$$

It is clear that the expression

$$XYZe^{-\kappa(\alpha^2+\beta^2+\gamma^2)t},$$

where

$$X=\cos\alpha x+\frac{h}{\alpha}\sin\alpha x,$$

$$Y=\cos\beta y+\frac{h}{\beta}\sin\beta y,$$

$$Z=\cos\gamma z+\frac{h}{\gamma}\sin\gamma z,$$

---

* Cf. Fourier, *loc. cit.*, Ch. VIII.

and a, β, γ are positive roots of

$$\left.\begin{aligned} \tan \alpha a &= \frac{2\alpha h}{\alpha^2-h^2}, \\ \tan \beta b &= \frac{2\beta h}{\beta^2-h^2}, \\ \tan \gamma c &= \frac{2\gamma h}{\gamma^2-h^2}, \end{aligned}\right\} \qquad (6)$$

satisfies (1), (2), (3) and (4). (Cf. § 36.)

Hence assuming the possibility of the expansion of the given arbitrary function in a triple series whose terms are of this nature, we have as the solution of our problem

$$v = \sum_{\alpha} \sum_{\beta} \sum_{\gamma} A_{\alpha,\beta,\gamma} X_\alpha Y_\beta Z_\gamma e^{-\kappa(\alpha^2+\beta^2+\gamma^2)t},$$

where $A_{\alpha,\beta,\gamma}$ is the coefficient of the term in $X_\alpha Y_\beta Z_\gamma$ in the expansion and the summation is taken over the infinite number of positive roots of the equations (6).

In the case when the initial temperature is constant and equal to v_0, this expression simplifies, as in § 50, and we have

$$\frac{v}{v_0} = 4^3 h^3 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{X_{2p+1}}{a(\alpha_{2p+1}^2+h^2)+2h} \, \frac{Y_{2q+1}}{b(\beta_{2q+1}^2+h^2)+2h}$$

$$\times \frac{Z_{2r+1}}{c(\gamma_{2r+1}^2+h^2)+2h} e^{-\kappa(\alpha_{2p+1}^2+\beta_{2q+1}^2+\gamma_{2r+1}^2)t}.$$

If t is so great that in each of these series the terms after the first may be neglected, we have

$$\frac{v}{v_0} = 4^3 h^3 \frac{X_1 Y_1 Z_1 e^{-\kappa(\alpha_1^2+\beta_1^2+\gamma_1^2)t}}{(a(\alpha_1^2+h^2)+2h)(b(\beta_1^2+h^2)+2h)(c(\gamma_1^2+h^2)+2h)}.$$

In applying this solution to the case of the cube, we have

$$\frac{v}{v_0} = \left[\frac{4h}{a(\alpha_1^2+h^2)+2h}\right]^3 X_1 Y_1 Z_1 e^{-3\kappa\alpha_1^2 t},$$

where
$$X_1 = \cos \alpha_1 x + \frac{h}{\alpha_1} \sin \alpha_1 x,$$

$$Y_1 = \cos \alpha_1 y + \frac{h}{\alpha_1} \sin \alpha_1 y,$$

and
$$Z_1 = \cos \alpha_1 z + \frac{h}{\alpha_1} \sin \alpha_1 z.$$

Therefore, if v_1 and v_2 are the temperatures at any point at t_1 and t

$$\frac{v_1}{v_2}=e^{2\kappa a_1^2(t_2-t_1)}.$$

Hence, κ having been found by other means, a_1 may be obtaine

But $$\tan\tfrac{1}{2}a_1 a=\frac{h}{a_1}.$$

Thus we have the value of h.

CHAPTER VII

THE FLOW OF HEAT IN A CIRCULAR CYLINDER

53. Introductory.

We have seen in § 6 that the equation of conduction, when expressed in cylindrical co-ordinates, becomes

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}\right).$$

If a circular cylinder whose axis coincides with the axis of z is heated, and the initial and boundary conditions are independent of the coordinates θ and z, the temperature will be a function of r and t only, and this equation reduces to

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right).$$

In this case the flow of heat takes place in planes perpendicular to the axis, and the lines of flow are radial.

When the initial and boundary conditions do not contain z, the flow of heat again takes place in planes perpendicular to the axis, and the equation of conduction reduces to

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right).$$

Again, when the initial and boundary conditions do not contain θ, the flow of heat takes place in planes through the axis, and the equation of conduction becomes

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}\right).$$

54. Infinite Cylinder. Steady Temperature.

If the solid is a hollow circular cylinder, whose inner and outer radii are r_1 and r_2, and the surfaces are kept at the constant temperatures v_1 and v_2, the equations for the temperature become

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}=0, \quad (r_1<r<r_2)$$

$$v=v_1, \text{ when } r=r_1,$$

and $$v=v_2, \text{ when } r=r_2.$$

Thus $$v=\frac{(v_2-v_1)\log r+v_1\log r_2-v_2\log r_1}{\log r_2-\log r_1}.$$

If the cylinder is solid and heat is supplied by an electric current of constant strength passing through a uniform straight wire coinciding with its axis, and the heating has gone on long enough for the steady state of temperature to be attained, the rate of flow of heat out through any concentric cylinder is equal to the rate at which heat is supplied to the wire by the current.

Thus, if H is the heat supplied per unit length per second, and K the Thermal Conductivity,

$$H=-2\pi rK\frac{dv}{dr}.$$

Therefore on integration we have

$$H\log\left(\frac{r_2}{r_1}\right)=2\pi K(v_1-v_2),$$

v_1 and v_2 being the temperatures at r_1 and r_2.

But $$H=I^2R,$$

I being the strength of the current and R the resistance.

Thus $$2\pi K(v_1-v_2)=I^2R\log\frac{r_2}{r_1}.*$$

55. Infinite Cylinder. Variable Temperature.

Let the initial temperature be given by $v=f(r)$ and let the surface $r=a$ be kept at a constant temperature, which may be taken as zero.†

* Cf. Niven, *London, Proc. R. Soc.*, 76 (A), p. 34, 1905; Lees, *London, Phil. Trans. R. Soc.*, 204 (A), p. 433, 1905.

† If the constant surface temperature is v_0, we may reduce this to the case of zero temperature by putting $v=v_0+w$.

The equations for v are as follows:

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right), \quad (0<r<a)$$

$$v=0, \quad \text{when } r=a,$$

and $$v=f(r), \quad \text{when } t=0.$$

If we put $v=e^{-\kappa\alpha^2 t}u$, where u is a function of r only, then we must have

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\alpha^2u=0,$$

which is Bessel's equation of order zero.

As the solution of the second kind is infinite at $r=0$, the particular integral of the temperature equation suitable for our problem is

$$v=AJ_0(\alpha r)\,e^{-\kappa\alpha^2 t},$$

where $J_0(x)$ is Bessel's function of order zero of the first kind.*

To satisfy the boundary condition α must be a root of

$$J_0(\alpha a)=0.$$

It is known that this equation has no imaginary roots or repeated roots, and that it has an infinite number of real positive roots

$$\alpha_1, \alpha_2, \alpha_3 \ldots .$$

Also to each positive root α there corresponds a negative root $-\alpha$.†

If $f(r)$ can be expanded in the series

$$A_1J_0(\alpha_1 r)+A_2J_0(\alpha_2 r)+\ldots,$$

the conditions of the problem will be satisfied by

$$v=\sum_1^\infty A_nJ_0(\alpha_n r)\,e^{-\kappa\alpha_n^2 t}.$$

Assuming for the present the possibility of the expansion‡ and that the series can be integrated term by term, we can obtain the values of the coefficients by the help of the two important definite integrals to be discussed in the next section.

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* For information as to the Bessel's functions see Appendix I.

† Cf. Gray and Mathews, *Bessel Functions*, Ch. V., 1895; Watson, *Theory of Bessel Functions*, Ch. XV., 1922. This important work has been passing through the press at the same time as my own. I am indebted to Professor Watson for the references to it here and in the pages which follow.

‡ For a discussion of the possibility of expanding an arbitrary function in a series of Bessel's functions, see Hobson, *London, Proc. Math. Soc.* (Ser. 2), 7, 1909; Moore, *Trans. Amer. Math. Soc.*, 10, 1909; 12, 1911; and 21, 1920; Young, *London, Proc. Math. Soc.* (Ser. 2), 18, 1920. The subject is also treated in Dini, *Serie di Fourier*, pp. 246-269, 1880, and Ford, *Studies in Divergent Series and Summability*, Ch. V., 1916.

56. The Integrals

$$\int_0^a rJ_n(\alpha r)J_n(\beta r)\,dr \quad \text{and} \quad \int_0^a r(J_n(\alpha r))^2\,dr.^*$$

Putting $u=J_n(\alpha r)$ and $v=J_n(\beta r)$, we have from Bessel's equation,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+\left(\alpha^2-\frac{n^2}{r^2}\right)u=0,$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)+\left(\beta^2-\frac{n^2}{r^2}\right)v=0.$$

Thus $$(\beta^2-\alpha^2)\int_0^a ruv\,dr=\int_0^a\left[v\frac{d}{dr}\left(r\frac{du}{dr}\right)-u\frac{d}{dr}\left(r\frac{dv}{dr}\right)\right]dr$$

$$=a\left[v\frac{du}{dr}-u\frac{dv}{dr}\right]_0^a,$$

and this vanishes when

$$aJ_n(\beta a)J_n'(\alpha a)-\beta J_n(\alpha a)J_n'(\beta a)=0,$$

where $$J_n'(\alpha a)=\left(\frac{d}{dr}J_n(r)\right)_{r=\alpha a}.$$

$$\left.\begin{array}{l}\text{Thus when } \alpha \text{ and } \beta \text{ are two different positive roots of}\\ \qquad\text{(i) } J_n(\alpha a)=0,\\ \qquad\text{or (ii) } J_n'(\alpha a)=0,\\ \qquad\text{or (iii) } \alpha J_n'(\alpha a)+hJ_n(\alpha a)=0,\text{—}\\ \text{we have}\qquad \displaystyle\int_0^a rJ_n(\alpha r)J_n(\beta r)\,dr=0.\end{array}\right\}\quad\text{..........(1)}$$

Again, since $$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+\left(\alpha^2-\frac{n^2}{r^2}\right)u=0,$$

$$2r\frac{du}{dr}\frac{d}{dr}\left(r\frac{du}{dr}\right)+2\left(\alpha^2-\frac{n^2}{r^2}\right)r^2u\frac{du}{dr}=0.$$

Therefore $$\frac{d}{dr}\left(r\frac{du}{dr}\right)^2+\alpha^2r^2\frac{du^2}{dr}-n^2\frac{du^2}{dr}=0,$$

and $$\alpha^2\int_0^a r^2\frac{du^2}{dr}\,dr+\left[\left(r\frac{du}{dr}\right)^2-n^2u^2\right]_0^a=0.$$

Integrate by parts and it follows that

$$2\alpha^2\int_0^a ru^2\,dr=\left[r^2\left(\frac{du}{dr}\right)^2+(\alpha^2r^2-n^2)u^2\right]_0^a.$$

---

* The convergence of the integrals when $r=0$ requires that the real part of n shall be greater than -1. In the applications of these integrals in the text we shall be dealing with n real and not less than zero.

Therefore

$$\int_0^a r(J_n(\alpha r))^2\,dr=\frac{1}{2\alpha^2}[a^2\alpha^2(J_n'(\alpha a))^2+(a^2\alpha^2-n^2)(J_n(\alpha a))^2]$$

$$=\frac{a^2}{2}\left[(J_n'(\alpha a))^2+\left(1-\frac{n^2}{a^2\alpha^2}\right)(J_n(\alpha a))^2\right].$$

Thus (i) when α is a root of $J_n(\alpha a)=0$,*

$$\int_0^a r(J_n(\alpha r))^2\,dr=\frac{a^2}{2}(J_n'(\alpha a))^2;$$

(ii) when α is a root of $J_n'(\alpha a)=0$,

$$\int_0^a r(J_n(\alpha r))^2\,dr=\frac{a^2}{2}\left(1-\frac{n^2}{a^2\alpha^2}\right)(J_n(\alpha a))^2;$$

and (iii) when α is a root of $\alpha J_n'(\alpha a)+hJ_n(\alpha a)=0$,

$$\int_0^a r(J_n(\alpha r))^2\,dr=\frac{1}{2\alpha^2}(a^2h^2+(a^2\alpha^2-n^2))(J_n(\alpha a))^2. \qquad \text{......(2)}$$

57. Infinite Cylinder (*continued*). Applications of these Integrals.

We may apply these results to the case of the Infinite Cylinder, assuming the possibility of the expansions needed in each case, and that they may be integrated term by term.†

I. *Surface $r=a$ at temperature zero. Initial temperature $v=f(r)$.*

In this case we take

$$f(r)=A_1J_0(\alpha_1 r)+A_2J_0(\alpha_2 r)+\dots,$$

where α_1, α_2 ... are the positive roots of

$$J_0(\alpha a)=0.$$

Thus $$A_n\int_0^a r(J_0(\alpha_n r))^2\,dr=\int_0^a rf(r)J_0(\alpha_n r)\,dr,$$

since $$\int_0^a rJ_0(\alpha_m r)J_0(\alpha_n r)\,dr=0.$$

But $$\int_0^a r(J_0(\alpha_n r))^2\,dr=\frac{a^2}{2}(J_0'(\alpha_n a))^2.$$

Therefore $$v=\frac{2}{a^2}\sum_1^\infty e^{-\kappa\alpha_n^2 t}\frac{\int_0^a rf(r)J_0(\alpha_n r)\,dr}{(J_0'(\alpha_n a))^2}J_0(\alpha_n r).$$

* It is known that the roots of this equation and the others in (ii) and (iii) are all real and not repeated. Cf. Watson, *loc. cit.*, §§ 15. 23, 15. 25.

† For a more rigorous treatment of some of the problems named in this section and the next, reference may be made to Moore's papers in *Trans. Amer. Math. Soc.*, 10, 1909; 12, 1911; and 14, 1913.

II. *Surface $r=a$ impervious to heat. Initial temperature $v=f(r)$.*

In this case we take

$$f(r)=A_1J_0(a_1r)+A_2J_0(a_2r)+\dots,$$

where a_1, a_2, ... are the positive roots of

$$J_0'(aa)=0.$$

Then, since $$\int_0^a r(J_0(a_nr))^2\,dr=\frac{a^2}{2}(J_0(a_na))^2,$$

we have $$v=\frac{2}{a^2}\sum_1^\infty e^{-\kappa a_n^2t}\frac{\int_0^a rf(r)J_0(a_nr)\,dr}{(J_0(a_na))^2}J_0(a_nr).$$

III. *Radiation at surface $r=a$ into a medium at zero temperature. Initial temperature $v=f(r)$.*

In this case we take

$$f(r)=A_1J_0(a_1r)+A_2J_0(a_2r)+\dots,$$

where a_1, a_2, ... are the positive roots of

$$aJ_0'(aa)+hJ_0(aa)=0.$$

Then, since $$\int_0^a r(J_0(a_nr))^2\,dr=\frac{a^2}{2a_n^2}(h^2+a_n^2)(J_0(a_na))^2,$$

we have $$v=\frac{2}{a^2}\sum_1^\infty e^{-\kappa a_n^2t}\frac{a_n^2\int_0^a rf(r)J_0(a_nr)\,dr}{(h^2+a_n^2)(J_0(a_na))^2}J_0(a_nr).$$

IV. *Surface $r=a$ at zero temperature. Initial temperature $v=f(r,\theta)$.*

In this case the equation of conduction becomes

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right),$$

and the expression $e^{-\kappa a^2t}J_n(ar)(A_n\cos n\theta+B_n\sin n\theta)$ satisfies this equation, n being taken integral as the temperature is periodic in θ with period 2π.

Now take the Fourier's Series for $f(r,\theta)$, namely,

$$f(r,\theta)=\sum_{n=0}^\infty(a_n\cos n\theta+b_n\sin n\theta),$$

where $$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(r,\theta)\cos n\theta\,d\theta,\quad (n\geqq 1)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(r,\theta)\sin n\theta\,d\theta,$$

and $$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(r,\theta)\,d\theta.$$

These coefficients are functions of r. Expand them in the series of Bessel's functions of the nth order, *e.g.*

$$a_n=\sum_{s=1}^{\infty}A_{n,s}J_n(a_s r),$$

where $a_1, a_2, \dots, a_s, \dots$, are the positive roots of

$$J_n(aa)=0.$$

Then we have

$$A_{0,s}=\frac{1}{\pi a^2(J_0'(aa))^2}\int_0^a\int_{-\pi}^{\pi}f(r,\theta)J_0(a_s r)r\,dr\,d\theta,$$

$$A_{n,s}=\frac{2}{\pi a^2(J_n'(a_s a))^2}\int_0^a\int_{-\pi}^{\pi}f(r,\theta)\cos n\theta\,J_n(a_s r)r\,dr\,d\theta,$$

and $$B_{n,s}=\frac{2}{\pi a^2(J_n'(a_s a))^2}\int_0^a\int_{-\pi}^{\pi}f(r,\theta)\sin n\theta\,J_n(a_s r)r\,dr\,d\theta.$$

Thus we obtain our solution in the form

$$v=\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(A_{n,s}\cos n\theta+B_{n,s}\sin n\theta)J_n(a_s r)e^{-\kappa a_s^2 t}.$$

V. *Radiation at surface $r=a$ into a medium at zero temperature. Initial temperature $v=f(r,\theta)$.*

In this case we take the Fourier's Series for $f(r,\theta)$, namely,

$$f(r,\theta)=\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$

as in IV.

The coefficients are functions of r. Expand them in the series of Bessel's functions of the nth order, *e.g.*

$$a_n=\sum_{s=1}^{\infty}A_{n,s}J_n(a_s r),$$

where $a_1, a_2, \dots$ are the positive roots of the equation

$$aJ_n'(aa)+hJ_n(aa)=0.$$

Then we have

$$A_{0,s}=\frac{a_s^2}{\pi a^2(a_s^2+h^2)(J_0(a_s a))^2}\int_0^a\int_{-\pi}^{\pi}f(r,\theta)J_0(a_s r)r\,dr\,d\theta,$$

$$A_{n,s}=\frac{2a_s^2}{\pi a^2\left(a_s^2+h^2-\dfrac{n^2}{a^2}\right)(J_n(a_s a))^2}\int_0^a\int_{-\pi}^{\pi}f(r,\theta)\cos n\theta J_n(a_s r)r\,dr\,d\theta,$$

$$B_{n,s}=\frac{2a_s^2}{\pi a^2\left(a_s^2+h^2-\dfrac{n^2}{a^2}\right)(J_n(a_s a))^2}\int_0^a\int_{-\pi}^{\pi}f(r,\theta)\sin n\theta J_n(a_s r)r\,dr\,d\theta,$$

and $$v=\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(A_{n,s}\cos n\theta+B_{n,s}\sin n\theta)J_n(a_s r)e^{-\kappa a_s^2 t}.$$

VI. *Surface $r=a$ at zero. Initial temperature $v=f(r, \theta, z)$.*

In this case we have

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

and $$e^{-\kappa(a^2+\mu^2)t}J_n(\mu r)\begin{matrix}\cos\\ \sin\end{matrix}n\theta\begin{matrix}\cos\\ \sin\end{matrix}az$$

is a particular integral.

Now expand $f(r, \theta, z)$ in the Fourier's Series

$$\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta).$$

The coefficients a_n and b_n are functions of r and z, denoted by $F_n(r, z)$ and $G_n(r, z)$.

Expand these functions in the series of Bessel's functions given by the positive roots of

$$J_n(\mu a)=0,$$

and let $$F_n(r, z)=\sum_{\mu}\phi_n(z)J_n(\mu r),$$

$$G_n(r, z)=\sum_{\mu}\psi_n(z)J_n(\mu r).$$

Finally, take the Fourier's Integrals for $\phi_n(z)$ and $\psi_n(z)$, namely,

$$\phi_n(z)=\frac{1}{\pi}\int_0^{\infty}da\int_{-\infty}^{\infty}\phi_n(\beta)\cos a(\beta-z)\,d\beta,$$

$$\psi_n(z)=\frac{1}{\pi}\int_0^{\infty}da\int_{-\infty}^{\infty}\psi_n(\beta)\cos a(\beta-z)\,d\beta.$$

Thus we get our solution in the form

$$v=\frac{1}{\pi}\sum_{\mu}\sum_{n=0}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}e^{-\kappa(a^2+\mu^2)t}J_n(\mu r)[\phi_n(\beta)\cos n\theta+\psi_n(\beta)\sin n\theta]\times\cos a(\beta-z)\,da\,d\beta,$$

the summation with regard to μ being over the positive roots of

$$J_n(\mu a)=0.$$

VII. *Surface $r=a$ at $v=F(\theta, z)$. Initial temperature $v=f(r, \theta, z)$.*

As shown in § 9, we reduce this to the case V., and a problem of steady temperature, by putting

$$v=u+w,$$

and choosing u, a function of r, θ, and z only, to satisfy

$$\left.\begin{aligned}&\nabla^2 u=0,\\ &u=F(\theta, z), \text{ when } r=a,\end{aligned}\right\}$$

and w to satisfy

$$\left.\begin{aligned}&\frac{\partial w}{\partial t}=\kappa\nabla^2 w,\\ &w=f(r, \theta, z)-u, \quad \text{when } t=0,\\ &w=0, \qquad\qquad\qquad \text{when } r=a.\end{aligned}\right\}$$

To find u we expand $F(\theta, z)$, as above, in the Fourier's Series

$$\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$

a_n and b_n being in this case functions of z, which we denote by $\phi_n(z)$ and $\psi_n(z)$. Then we take the Fourier's Integrals for $\phi_n(z)$ and $\psi_n(z)$, and u is given by the equation

$$u=\frac{1}{\pi}\sum_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{J_n(\iota\alpha r)}{J_n(\iota\alpha a)}(\phi_n(\beta)\cos n\theta+\psi_n(\beta)\sin n\theta)$$
$$\times\cos\alpha(\beta-z)\,d\alpha\,d\beta.$$

To find w we proceed as in VI. above.

58. Semi-Infinite Cylinder. Steady Temperature.

Let the axis of the cylinder as before lie along the axis of z and let its base be the plane $z=0$. We shall examine first of all the case of steady temperature when the base is kept at temperature $v=f(r)$ and radiation takes place into a medium at zero temperature at the surface $r=a$. When we put $f(r)=v_0$ the solution of this problem will correspond to the exact discussion of the Flow of Heat in a Rod which has been treated in § 20, with the assumption that the cross-section is so small that the temperature over it may be taken as equal to that at its centre. However, when radiation takes place at the surface, there must be a flow of heat outwards from the middle of the rod, and the assumption of Linear Flow of Heat serves only as an approximation to the actual state of affairs. This approximation is admissible when the emissivity is poor and the conductivity good.* For this reason the Bar Methods of determining the Conductivity are employed only for good conductors. We have seen in § 52 that in the case of poor

* Cf. Peck, *Phil. Mag., London* (Ser. 6), 4, 1902.

conductors experiments have been conducted on cubical blo of the substance, and in §§ 63, 66 we shall find that cylinders a spheres may be employed for the same purpose.

In this case of steady temperature the equations for v are

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial z^2}=0, \quad (0<r<a;\ 0<z) \quad \ldots\ldots\ldots$$

$$v=f(r), \text{ when } z=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial v}{\partial r}+hv=0, \text{ when } r=a. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The expression $\qquad e^{-az}J_0(ar)$

is a particular integral of (1) and satisfies (3) if

$$aJ_0'(aa)+hJ_0(aa)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

Thus we take $\quad f(r)=A_1J_0(a_1r)+A_2J_0(a_2r)+\ldots,$

where $A_n=\dfrac{2a_n^2}{a^2(h^2+a_n^2)(J_0(a_na))^2}\displaystyle\int_0^a rf(r)J_0(a_nr)\,dr$ (cf. § 56 (2))

and $\qquad v=\displaystyle\sum_1^\infty A_ne^{-a_nz}J_0(a_nr).$

When a is small, the roots $a_1, a_2, \ldots$ increase rapidly, and we may take f v the first term in this expansion.*

Further, if $a_1^2a^2$ may be neglected, we have

$$J_0(a_1a)=1 \quad \text{and} \quad J_0'(a_1a)=-\tfrac{1}{2}a_1a.$$

Therefore, from (4), $\qquad -\tfrac{1}{2}a_1^2a+h=0,$

or $\qquad a_1=\sqrt{\left(\dfrac{2h}{a}\right)}.$

It will be noticed that this requires ah to be small, and that we obtain f A_1 to this approximation the value V when $f(r)$ is constant and equal to V

Thus we have $\qquad v=Ve^{-\sqrt{\left(\frac{2h}{a}\right)}z},$

and this agrees with the solution of § 20.

59. Semi-Infinite Cylinder. Variable Temperature.

We shall examine first the problem of the Semi-Infinite Cylinde given that

$$v=f(r,\theta,z) \text{ initially,}$$

and $\qquad v=0, \text{ when } r=a \text{ and } z=0.$

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* Cf. Watson, *loc. cit.*, §§ 15. 23, 15. 32; Moore, *Trans. Amer. Math. So* 10, p. 397, 1909.

In this case we start with the expression

$$e^{-\kappa(a^2+\mu^2)t} J_n(\mu r) \frac{\cos}{\sin} n\theta \sin az,$$

which satisfies
$$\frac{\partial v}{\partial t} = \kappa \nabla^2 v,$$

and also the surface conditions when $z=0$ and $r=a$, if $J_n(\mu a)=0$.

Now expand $f(r, \theta, z)$ in the Fourier's Series

$$\sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

a_n and b_n being functions of r and z, denoted by $F_n(r, z)$ and $G_n(r, z)$.

Then expand $F_n(r, z)$ and $G_n(r, z)$ in the series of Bessel's functions given by the positive roots of

$$J_n(\mu a)=0,$$

and take Fourier's Sine Integrals for the coefficients $\phi_n(z)$ and $\psi_n(z)$ of the terms in these expansions.

In this way we find the solution in the form

$$v = \frac{2}{\pi} \sum_{\mu} \sum_{n=0}^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-\kappa(a^2+\mu^2)t} J_n(\mu r)[\phi_n(\beta) \cos n\theta + \psi_n(\beta) \sin n\theta]$$
$$\times \sin a\beta \sin az \, da \, d\beta,$$

the summation with regard to μ being over the positive roots of

$$J_n(\mu a)=0.$$

If the temperature at the surface is given by

$$v = \chi_1(r, \theta), \text{ when } z=0,$$
$$v = \chi_2(\theta, z), \text{ when } r=a,$$

and if
$$v = f(r, \theta, z), \text{ when } t=0,$$

we proceed as in § 9 to break up the problem into two, the one being a case of steady temperature, and the other a case of variable temperature. In the steady temperature problem we have

$$\nabla^2 u = 0,$$
$$u = \chi_1(r, \theta), \text{ when } z=0,$$
$$u = \chi_2(\theta, z), \text{ when } r=a.$$

Then we put $u = u_1 + u_2$, and choose u_1, u_2 to satisfy

$$\nabla^2 u_1 = 0,$$
$$u_1 = \chi_1(r, \theta), \text{ when } z=0,$$

and
$$\nabla^2 u_2 = 0,$$
$$u_2 = 0, \text{ when } z=0,$$
$$u_2 = \chi_2(z, \theta) - u_1, \text{ when } r=a.$$

Thus we have*

$$u_1 = \frac{1}{2\pi}\int_0^a r'\,dr'\int_{-\pi}^{\pi} \chi_1(r', \theta')\,d\theta'\int_0^\infty e^{-\lambda z} J_0(\lambda R)\,d\lambda$$

and $u_2 = \dfrac{2}{\pi}\displaystyle\int_0^\infty \sin\lambda z \frac{J_n(i\lambda r)}{J_n(i\lambda a)}\,d\lambda\int_0^\infty [\phi_n(\alpha)\cos n\theta + \psi_n(\alpha)\sin n\theta]\sin\lambda\alpha\,d\alpha,$

where $R = \sqrt{(r^2 + r'^2 - 2rr'\cos(\theta - \theta'))}$ and $\phi_n(z)$ and $\psi_n(z)$ are the coefficients in the Fourier's Series

$$\sum_{n=0}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

for

$$\chi_2(z, \theta) - u_1.$$

The variable temperature problem is the same as that discussed in the former part of this article, the initial temperature being

$$f(r, \theta, z) - u.$$

60. Finite Cylinder. Surface at Zero Temperature. Initial Temperature f(r, θ, z).

The equations for the temperature are

$$\frac{\partial v}{\partial t} = \kappa\nabla^2 v, \quad (0 < r < a, \ -l < z < l), \quad \text{..............(1)}$$

$$v = f(r, \theta, z), \text{ when } t = 0, \quad \text{..........................(2)}$$

and

$$v = 0, \text{ when } r = a \text{ and } z = \pm l. \quad \text{........................(3)}$$

The expression

$$e^{-\kappa\left(\mu^2 + \frac{m^2\pi^2}{4l^2}\right)t} J_n(\mu r)\frac{\cos}{\sin} n\theta \sin\frac{m\pi}{2l}(z + l)$$

satisfies (1) and (3), if m is any integer and μ is a root of

$$J_n(\mu a) = 0.$$

Now expand $f(r, \theta, z)$ in the Fourier's Series

$$\sum_{n=0}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$

a_n and b_n being functions of r and z, denoted by $F_n(r, z)$ and $G_n(r, z)$. Then expand $F_n(r, z)$ and $G_n(r, z)$ in the series of Bessel's functions given by the positive roots of $J_n(\mu a) = 0$, and take the Sine Series, whose terms are the sines of multiples of $\dfrac{\pi}{2l}(z + l)$, for the coefficients in this series.

In this way we find the solution in the form

$$v = \sum_{\mu}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} e^{-\kappa\left(\mu^2 + \frac{m^2\pi^2}{4l^2}\right)t} J_n(\mu r)\sin\frac{m\pi}{2l}(z + l)(A_{\mu, m, n}\cos n\theta + B_{\mu, m, n}\sin n\theta),$$

* Heine, *Handbuch der Kugelfunctionen* (2. Aufl.), Bd. II., § 57.

where

$$A_{\mu,m,n}=\frac{2\int_0^a rJ_n(\mu r)\,dr\int_{-l}^{l}\sin\frac{m\pi}{2l}(z+l)\,dz\int_{-\pi}^{\pi}\cos n\theta f(r,\theta,z)\,d\theta}{\pi a^2 l(J_n'(\mu a))^2},$$

and a similar expression holds for $B_{\mu,m,n}$.*

If the temperature at the surface is given by

$$v=\chi_1(\theta,z),\ \text{when } r=a,$$
$$v=\chi_2(r,\theta),\ \text{when } z=l,$$
$$v=\chi_3(r,\theta),\ \text{when } z=-l,$$

we have, as before, to consider the steady temperature problem

$$\nabla^2 u=0,$$

where

$$u=\chi_1(\theta,z),\ \text{when } r=a,$$
$$u=\chi_2(r,\theta),\ \text{when } z=l,$$

and

$$u=\chi_3(r,\theta),\ \text{when } z=-l.$$

This may be solved by taking

$$u=u_1+u_2+u_3,$$

where u_1 satisfies $\nabla^2 u_1=0$, has the given value at $r=a$, and is zero at $z=\pm l$, with similar conditions for the others.

In this way we obtain

$$u_1=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{J_n\left(i\frac{m\pi}{l}r\right)}{J_n\left(i\frac{m\pi}{l}a\right)}\sin\frac{m\pi}{2l}(z+l)(A_{m,n}\cos n\theta+B_{m,n}\sin n\theta),$$

where $A_{m,n}$ and $B_{m,n}$ are determined by the expansion of $\chi_1(\theta,z)$ as above.

Also
$$u_2=\sum_{\mu}\sum_{n=0}^{\infty}\frac{\sinh\mu(z+l)}{\sinh 2\mu l}J_n(\mu r)(A_{\mu,n}\cos n\theta+B_{\mu,n}\sin n\theta),$$

where the summation in μ is over the positive roots of

$$J_n(\mu a)=0$$

and $A_{\mu,n}$, $B_{\mu,n}$ are determined by the expansion of $\chi_2(r,\theta)$.

Similarly

$$u_3=\sum_{\mu}\sum_{n=0}^{\infty}\frac{\sinh\mu(l-z)}{\sinh 2\mu l}J_n(\mu r)(A_{\mu,n}\cos n\theta+B_{\mu,n}\sin n\theta),$$

where $A_{\mu,n}$, $B_{\mu,n}$ are determined by the expansion of $\chi_3(r,\theta)$.

61. Finite Cylinder. Radiation.

If, in the cylinder of § 60, radiation takes place at all the surfaces into a medium at zero temperature, and there is an arbitrary initial

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* Cf. Heine, *loc. cit.*, Bd. II., § 81; when $n=0$, the expression is to be halved.

temperature $v=f(r, \theta, z)$, we have the following equations for the temperature:

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v, \quad \text{.......................................(1)}$$

$$v=f(r, \theta, z), \text{ when } t=0, \quad \text{.................(2)}$$

$$\frac{\partial v}{\partial r}+hv=0, \qquad \text{when } r=a, \quad \text{.................(3)}$$

$$\frac{\partial v}{\partial z}+hv=0, \qquad \text{when } z=l, \quad \text{.................(4)}$$

and

$$-\frac{\partial v}{\partial z}+hv=0, \qquad \text{when } z=-l. \quad \text{...............(5)}$$

Put $v=u+w$, where u, w satisfy (1), (3), (4), and (5), and

$$u=\tfrac{1}{2}\{f(r, \theta, z)-f(r, \theta, -z)\}, \text{ when } t=0,$$

and

$$w=\tfrac{1}{2}\{f(r, \theta, z)+f(r, \theta, -z)\}, \text{ when } t=0.$$

To determine u we take the expression

$$\sin\lambda z \,{}^{\cos}_{\sin}\, n\theta\, J_n(\mu r)e^{-\kappa(\lambda^2+\mu^2)t}.$$

This satisfies (1) and (3), if μ is a root of

$$\mu J_n'(\mu a)+hJ_n(\mu a)=0. \quad \text{.........................(6)}$$

Further, it satisfies (4) and (5), if λ is a root of

$$\lambda\cos\lambda l+h\sin\lambda l=0. \quad \text{.........................(7)}$$

If we then expand the function

$$\tfrac{1}{2}\{f(r, \theta, z)-f(r, \theta, -z)\}$$

in the series,

$$\sum_\lambda\sum_\mu\sum_{n=0}^{\infty}(A_{\lambda,\mu,n}\cos n\theta+B_{\lambda,\mu,n}\sin n\theta)\sin\lambda z\, J_n(\mu r),$$

u is given by the equation

$$u=\sum_\lambda\sum_\mu\sum_{n=0}^{\infty}(A_{\lambda,\mu,n}\cos n\theta+B_{\lambda,\mu,n}\sin n\theta)\sin\lambda z J_n(\mu r)e^{-\kappa(\lambda^2+\mu^2)t},$$

the summations with regard to μ and λ being taken over the positive roots of (6) and (7) respectively.

To determine w, we take the expression

$$\cos\lambda z \,{}^{\cos}_{\sin}\, n\theta J_n(\mu r)e^{-\kappa(\lambda^2+\mu^2)t},$$

and the λ's are the positive roots of

$$h\cos\lambda l - \lambda\sin\lambda l = 0.$$

The function $\quad \frac{1}{2}\{f(r, \theta, z) + f(r, \theta, -z)\}$

is then expanded in a series whose terms are of this type, and v is given by the equation *

$$v = \sum_{\lambda}\sum_{\mu}\sum_{n=0}^{\infty}(A_{\lambda,\mu,n}\cos n\theta + B_{\lambda,\mu,n}\sin n\theta)\cos\lambda z\, J_n(\mu r)e^{-\kappa(\lambda^2+\mu^2)t}.$$

62. More General Problems on the Cylinder.

The methods of the preceding sections may be used in dealing with the hollow cylinder, or the solid in which the bounding surface is formed by the cylinder (or hollow cylinder), two planes through the axis, and one or two planes perpendicular to the axis.

It will be sufficient to give here the solutions of the following problems of this kind:

I. *Infinite Hollow Cylinder. Surfaces $r=a$ and $r=b$ kept at zero. Initial Temperature $f(r)$.*

In this case we have

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right). \quad \text{.........................(1)}$$

If we put $v = e^{-\kappa\alpha^2 t}u$, where u depends on r only, the equation for u is

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \alpha^2 u = 0,$$

or Bessel's equation of order zero.

As the range of r does not extend to the origin, Bessel's functions of the second kind are not excluded. Instead of introducing the function Y_n (cf. Appendix I., § 2), it is better to take the function $H_n^{(1)}$, where $H_n^{(1)} = J_n + iY_n$ (cf. Appendix I., § 4), since $H_n^{(1)}(z)$ vanishes at infinity in the upper part of the z-plane.

Let $\quad U_0(\alpha r) = J_0(\alpha r)H_0^{(1)}(\alpha b) - J_0(\alpha b)H_0^{(1)}(\alpha r), \quad$ ....(2)

where $H_0^{(1)}(r)$ is this solution of Bessel's equation of order zero.

Then $U_0(\alpha b) = 0$ and $U_0(\alpha a)$ is also zero, provided that α is a root of

$$J_0(\alpha a)H_0^{(1)}(\alpha b) - J_0(\alpha b)H_0^{(1)}(\alpha a) = 0. \quad \text{...............(3)}$$

This equation has no imaginary or repeated roots, and it has

* Cf. Heine, *loc. cit.*, Bd. II., § 83.

an infinite number of real roots. To every positive root α there corresponds a negative root $-\alpha$.*

Further, we have, as in § 56,

$$\int_a^b rU_0(\alpha r)\,U_0(\beta r)\,dr = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4$$

where α, β are two different positive roots of (3).

And $$\int_a^b rU_0^2(\alpha r)\,dr = \frac{1}{2\alpha^2}\left[\left(r\frac{dU_0}{dr}\right)^2\right]_a^b.$$

But

$$\left(r\frac{dU_0}{dr}\right)_{r=b} = \alpha b\left\{H_0^{(1)}(\alpha b)\frac{d}{d(\alpha b)}J_0(\alpha b) - J_0(\alpha b)\frac{d}{d(\alpha b)}H_0^{(1)}(\alpha b)\right\}$$
$$= -\frac{2i}{\pi},$$

since $$J_0(z)\frac{d}{dz}H_0^{(1)}(z) - H_0^{(1)}(z)\frac{d}{dz}J_0(z) = \frac{2i}{\pi z}.\dagger$$

---

*The equation $U_0(\alpha a)=0$ is the same as

$$J_0(\alpha a)\,J_0(\alpha b)\left\{\frac{Y_0(\alpha b)}{J_0(\alpha b)} - \frac{Y_0(\alpha a)}{J_0(\alpha a)}\right\} = 0.$$

The real roots of this equation are known [cf. Gray and Mathews, *loc. cit.*, p. 242 (vi)].

To show that there are no pure imaginary roots, we see from Appendix I., § that

$$\pi Y_0(z) = 2J_0(z)(\log z/2+\gamma) + (z/2)^2 - \frac{1+\frac{1}{2}}{(2!)^2}(z/2)^4 + \ldots.$$

Thus $$y = \frac{Y_0(ix)}{J_0(ix)} - i$$

is real, when x is real.

Also $$\frac{dy}{dx} = \frac{i}{J_0^2(ix)}\left\{J_0(z)\,Y_0'(z) - J_0'(z)\,Y_0(z)\right\}_{z=ix}$$
$$= \frac{2}{\pi x J_0^2(ix)} \text{ [cf. Watson, \textit{loc. cit.}, § 3. 63 (1)].}$$

Therefore y is a continually increasing function of x.

It follows that $$\frac{Y_0(\alpha b)}{J_0(\alpha b)} - \frac{Y_0(\alpha a)}{J_0(\alpha a)} = 0$$

cannot have any pure imaginary root.

Again, we know that

$$(\alpha^2-\beta^2)\int_a^b rU_0(\alpha r)\,U_0(\beta r)\,dr = 0,$$

when α, β are different roots of $U_0(\alpha a)=0$.

Thus $U_0(\alpha a)=0$ cannot have an imaginary root of the form $\lambda\pm$ (cf. § 36, p. 78).

The equation $U_n(\alpha a)=0$ can be treated in the same way.

† Cf. Watson, *loc. cit.*, § 3. 63 (1).

Also

$$\left(r\frac{dU_0}{dr}\right)_{r=a}=\alpha a\left\{H_0^{(1)}(\alpha b)\frac{d}{d(\alpha a)}J_0(\alpha a)-J_0(\alpha b)\frac{d}{d(\alpha a)}H_0^{(1)}(\alpha a)\right\}.$$

But
$$\frac{J_0(\alpha a)}{J_0(\alpha b)}=\frac{H_0^{(1)}(\alpha a)}{H_0^{(1)}(\alpha b)}=\rho, \text{ say.}$$

It follows that
$$\left(r\frac{dU_0}{dr}\right)_{r=a}=-\frac{2i}{\rho\pi}.$$

Therefore we have

$$\int_a^b rU_0^2(\alpha r)\,dr=\frac{2}{\pi^2\alpha^2}\left\{\frac{J_0^2(\alpha b)-J_0^2(\alpha a)}{J_0^2(\alpha a)}\right\}. \quad \text{..............(5)}$$

Assuming that $f(r)$ can be expanded in the series

$$A_1U_0(\alpha_1 r)+A_2U_0(\alpha_2 r)+\dots,$$

and that the series can be integrated term by term, we have, from (4) and (5),

$$A_n=\frac{\int_a^b rf(r)\,U_0(\alpha_n r)\,dr}{\int_a^b rU_0^2(\alpha_n r)\,dr}$$

$$=\frac{\pi^2\alpha_n^2}{2}\,\frac{J_0^2(\alpha_n a)}{J_0^2(\alpha_n b)-J_0^2(\alpha_n a)}\int_a^b rf(r)\,U_0(\alpha_n r)\,dr.$$

Thus we are led to the solution of our problem in the form

$$v=\frac{\pi^2}{2}\sum_1^\infty \alpha_n^2\,\frac{J_0^2(\alpha_n a)}{J_0^2(\alpha_n b)-J_0^2(\alpha_n a)}\,e^{-\kappa\alpha_n^2 t}U_0(\alpha_n r)\int_a^b rf(r)\,U_0(\alpha_n r)\,dr,$$

the summation being taken over the positive roots of (3).

II. *Infinite Cylinder. The surface $r=a$ and the planes $\theta=0$, $\theta=\theta_0$ kept at zero. Initial temperature $f(r, \theta)$.*

In this case we have

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}\right),$$

and
$$J_{\frac{m\pi}{\theta_0}}(\alpha r)\sin\frac{m\pi}{\theta_0}\theta\, e^{-\kappa\alpha^2 t}$$

is a particular integral of this equation.

Also the conditions at $r=a$, $\theta=0$ and $\theta=\theta_0$ are satisfied, provided that m is a positive integer and α is a root of

$$J_{\frac{m\pi}{\theta_0}}(\alpha a)=0.$$

Expand $f(r, \theta)$ in the Sine Series

$$\sum_1^\infty a_m \sin \frac{m\pi}{\theta_0}\theta,$$

the coefficient a_m being a function of r, say $F_m(r)$.

Then expand $F_m(r)$ in the series of Bessel's functions given by the positive roots of

$$J_{\frac{m\pi}{\theta_0}}(\alpha a)=0.$$

In this way we are brought to the solution of our problem in the form

$$v=\sum_\alpha \sum_{m=1}^\infty A_{\alpha, m} e^{-\kappa\alpha^2 t} \sin \frac{m\pi}{\theta_0}\theta\, J_{\frac{m\pi}{\theta_0}}(\alpha r),$$

where

$$A_{\alpha, m}=\frac{2}{\theta_0 \int_0^a r J^2_{\frac{m\pi}{\theta_0}}(\alpha r)\, dr}\int_0^a\int_0^{\theta_0} f(r, \theta) \sin \frac{m\pi}{\theta_0}\theta J_{\frac{m\pi}{\theta_0}}(\alpha r) r\, dr\, d\theta$$

$$=\frac{4}{a^2\theta_0 (J'_{\frac{m\pi}{\theta_0}}(\alpha r))^2}\int_0^a\int_0^{\theta_0} f(r, \theta) \sin \frac{m\pi}{\theta_0}\theta\, J_{\frac{m\pi}{\theta_0}}(\alpha r)\, r\, dr\, d\theta,$$

the summation in α being over the positive roots of

$$J_{\frac{m\pi}{\theta_0}}(\alpha a)=0.$$

The solution for the wedge given by $\theta=0$ and $\theta=\theta_0$ can be deduced from the above by letting $a \to \infty$. Cf. §§ 69, 90.

III. *Finite Cylinder. The ends $z=\pm l$, the planes $\theta=0$, $\theta=\theta_0$ and the surface $r=a$ kept at zero. Initial temperature $f(r, \theta, z)$.*

In this case we have

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

and a particular integral of this equation is

$$e^{-\kappa\left(\lambda^2+\frac{m^2\pi^2}{4l^2}\right)t} J_{\frac{n\pi}{\theta_0}}(\lambda r) \sin \frac{n\pi}{\theta_0}\theta \sin \frac{m\pi}{2l}(z+l).$$

Also the conditions at the surface are satisfied if m, n are positive integers, and λ is a root of $J_{\frac{n\pi}{\theta_0}}(\lambda a)=0.$

Expand $f(r, \theta, z)$ in the Sine Series

$$\sum_1^\infty a_n \sin \frac{n\pi}{\theta_0}\theta,$$

a_n being a function of r and z, say $F_n(r, z)$.

Then expand $F_n(r, z)$ in the Sine Series whose terms are multiples of $\frac{\pi}{2l}(z+l)$. The coefficients of this series will be functions of r, say $F_{m,n}(r)$.

Finally expand $F_{m,n}(r)$ in the series of Bessel's functions given by the positive roots of $J_{\frac{n\pi}{\theta_0}}(\lambda a)=0.$

In this way we are led to the solution of our problem in the form

$$v=\sum_\lambda \sum_m \sum_n A_{\lambda,m,n} e^{-\kappa\left(\lambda^2+\frac{m^2\pi^2}{4l^2}\right)t} J_{\frac{n\pi}{\theta_0}}(\lambda r) \sin\frac{m\pi}{2l}(z+l)\sin\frac{n\pi}{\theta_0}\theta,$$

where $A_{\lambda,m,n}=\dfrac{4}{a^2 l\theta_0 (J'_{\frac{n\pi}{\theta_0}}(\lambda a))^2}$

$$\times\int_0^a\int_0^{\theta_0}\int_{-l}^{l} f(r,\theta,z)\sin\frac{m\pi}{2l}(z+l)\sin\frac{n\pi}{\theta_0}\theta\, J_{\frac{n\pi}{\theta_0}}(\lambda r)\, r\,dr\,d\theta\,dz.$$

63. Determination of the Conductivity from Cylinders.

The results of the last sections can be reduced to a simpler form when the initial temperature is constant. We proceed to examine three cases which lend themselves to experimental investigation.

I. *Initial temperature* $v=v_0$. *Radiation at* $r=a$, $z=\pm l$ *into a medium at zero.**

In this case we take the expression

$$\cos\lambda z J_0(\mu r)e^{-\kappa(\lambda^2+\mu^2)t},$$

which is a particular integral of the equation

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v.$$

This satisfies the surface conditions, if λ is a positive root of the equation

$$h\cos l\lambda-\lambda\sin l\lambda=0$$

and μ is a positive root of $\mu J_0'(\mu a)+hJ_0(\mu a)=0.$

Now, assuming as before the possibility of the expansions

$$1=A_1\cos\lambda_1 z+A_2\cos\lambda_2 z+\dots,$$
$$1=B_1 J_0(\mu_1 r)+B_2 J_0(\mu_2 r)+\dots,$$

we may obtain the values of these coefficients by integration.

---

* Weber, H. F., *Ann. Physik, Leipzig* (N.F.), 10, p. 111, 1880.

To determine A_1, A_2, ... we first show that

$$\int_{-l}^{l} \cos\lambda_m z \cos\lambda_n z\, dz = 0 \quad (m \neq n),$$

and

$$\int_{-l}^{l} \cos^2\lambda_n z\, dz = \frac{4\sin\lambda_n l}{2\lambda_n l + \sin 2\lambda_n l}.$$

We have

$$\int_{-l}^{l} \cos\lambda_m z \cos\lambda_n z\, dz$$

$$= \tfrac{1}{2}\int_{-l}^{l} (\cos(\lambda_m+\lambda_n)z + \cos(\lambda_m-\lambda_n)z)\, dz$$

$$= \frac{\sin(\lambda_m+\lambda_n)l}{2(\lambda_m+\lambda_n)} + \frac{\sin(\lambda_m-\lambda_n)l}{2(\lambda_m-\lambda_n)}$$

$$= \frac{\cos\lambda_m l \cos\lambda_n l}{(\lambda_m^2-\lambda_n^2)} \quad (\lambda_m \tan\lambda_m l - \lambda_n \tan\lambda_n l)$$

$$= 0, \text{ since } \lambda\tan\lambda l = h.$$

Also

$$\int_{-l}^{l} \cos^2\lambda_n z\, dz = \tfrac{1}{2}\int_{-l}^{l} (1+\cos 2\lambda_n z)\, dz$$

$$= l + \frac{1}{2\lambda_n}\sin 2\lambda_n l.$$

Therefore, on multiplying by $\cos\lambda_n z$ and integrating, we have

$$A_n = \frac{4\sin\lambda_n l}{2\lambda_n l + \sin 2\lambda_n l}.$$

To determine B_1, B_2, ... we have

$$B_n\int_0^a rJ_0^2(\mu_n r)\, dr = \int_0^a rJ_0(\mu_n r)\, dr.$$

But since $\dfrac{d^2}{dr^2}J_0(\mu_n r) + \dfrac{1}{r}\dfrac{d}{dr}J_0(\mu_n r) + \mu^2 J_0(\mu_n r) = 0,$

we have

$$\mu_n^2\int_0^a rJ_0(\mu_n r)\, dr = -\int_0^a \frac{d}{dr}\left(r\frac{d}{dr}J_0(\mu_n r)\right) dr$$

$$= -\mu_n a J_0'(\mu_n a) = ahJ_0(\mu_n a).$$

Also we have seen (§ 56) that

$$\int_0^a rJ_0^2(\mu_n r)\, dr = \frac{a^2}{2\mu_n^2}(h^2+\mu_n^2)J_0^2(\mu_n a).$$

Therefore

$$B_n = \frac{2h}{a(h^2+\mu_n^2)J_0(\mu_n a)}.$$

Thus we have the solution in the form

$$v=v_0(A_1 \cos \lambda_1 z\, e^{-\kappa\lambda_1^2 t}+\ldots)(B_1 J_0(\mu_1 r)e^{-\kappa\mu_1^2 t}+\ldots),$$

$\lambda_1, \lambda_2, \ldots$, and $\mu_1, \mu_2, \ldots$ being the positive roots of the equations

$$h \cos l\lambda - \lambda \sin l\lambda = 0 \quad \text{and} \quad \mu J_0'(\mu a)+hJ_0(\mu a)=0.$$

Further, if we take temperature readings after a considerable time, we have approximately

$$v=v_0 A_1 B_1 \cos \lambda_1 z J_0(\mu_1 r)e^{-\kappa(\lambda_1^2+\mu_1^2)t}.$$

Two observations v_1, v_2, at the same point at times t_1, t_2, give

$$\frac{v_1}{v_2}=e^{\kappa(\lambda_1^2+\mu_1^2)(t_2-t_1)};$$

and κ may be determined, h being supposed known.

II. *Initial temperature $v=v_0$. Radiation at $z=\pm l$ into a medium at zero. Surface $r=a$ kept at zero.**

In this case we have, as before,

$$v=v_0(A_1 \cos \lambda_1 z\, e^{-\kappa\lambda_1^2 t}+\ldots)(B_1 J_0(\mu_1 r)e^{-\kappa\mu_1^2 t}+\ldots),$$

where $\lambda_1, \lambda_2, \ldots$ are the positive roots of

$$h \cos l\lambda - \lambda \sin l\lambda = 0,$$

and $\mu_1, \mu_2, \ldots$ are the positive roots of $J_0(\mu a)=0$.

Also $$A_n=\frac{4 \sin \lambda_n l}{2\lambda_n l+\sin 2\lambda_n l}$$

and $$B_n=-\frac{2}{a\mu_n J_0'(\mu_n a)}.$$

Approximating as before,

$$v=v_0 A_1 B_1 \cos \lambda_1 z J_0(\mu_1 r)e^{-\kappa(\lambda_1^2+\mu_1^2)t},$$

and from two observations at the same point we obtain the value of κ, h being supposed known.

III. *Initial temperature $v=v_0$. Surface $z=-l$ kept at zero temperature. Radiation at $z=+l$ and $r=a$ into a medium at zero.*†

In this case we start with the particular integral

$$\sin \lambda(z+l)J_0(\mu r)e^{-\kappa(\lambda^2+\mu^2)t}.$$

* Weber, H. F., *Berlin, SitzBer. Ak. Wiss.*, p. 472, 1880.

† Beglinger, *Berlin, Verh. Ver. Gewerbfl.*, 45, 1896. See also Hall, *Physic. Rev., Ithaca, N.Y.*, 10, p. 297, 1900.

The surface conditions are satisfied if λ and μ are given by the equations

$$\lambda \cos 2\lambda l + h \sin 2\lambda l = 0$$

and

$$\mu J_0'(\mu a) + h J_0(\mu a) = 0.$$

Thus we have the solution in the form

$$v = v_0 (A_1 \sin \lambda_1 (z+l) e^{-\kappa\lambda_1^2 t} + \ldots)(B_1 J_0(\mu_1 r) e^{-\kappa\mu_1^2 t} + \ldots),$$

where

$$A_n = \frac{4(1-\cos 2\lambda_n l)}{4\lambda_n l - \sin 4\lambda_n l},$$

$$B_n = \frac{2h}{a(h^2+\mu_n^2) J_0(\mu_n a)},$$

and $\lambda_1, \lambda_2, \ldots; \mu_1, \mu_2, \ldots$ are the positive roots of the above equations.

Approximating as before,

$$v = v_0 A_1 B_1 \sin \lambda_1 (z+l) J_0(\mu_1 r) e^{-\kappa(\lambda_1^2+\mu_1^2)t}.$$

CHAPTER VIII

THE FLOW OF HEAT IN A SPHERE AND CONE

64. Introductory.

We have seen in § 6 that the equation of conduction, when expressed in spherical polar coordinates, becomes

$$\frac{\partial v}{\partial t}=\kappa\left\{\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}\right\}.$$

In the case of Flow of Heat in the Sphere, when the initial and surface conditions are such that the isothermal surfaces are concentric spheres, and the temperature thus depends only upon the coordinates r and t, this equation becomes

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}\right).$$

On putting $u=vr$, we have

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial r^2}.$$

Steady Temperature.

If the solid is a hollow sphere, of inner radius r_1 and outer radius r_2, we have

$$u=vr,$$

where

$$\frac{d^2u}{dr^2}=0,$$

and

$$u=v_1r_1 \text{ at } r=r_1,$$
$$u=v_2r_2 \text{ at } r=r_2,$$

the temperatures of the inner and outer surfaces being v_1 and v_2.

Therefore

$$v=\frac{v_1r_1(r_2-r)+v_2r_2(r-r_1)}{r(r_2-r_1)}.$$

Variable Temperature.

Let the sphere be of radius a, and the initial temperature be given by $v=f(r)$. If the surface is kept at a constant temperature v_0, the equations for u are as follows:

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial r^2}, \quad (0<r<a)$$

$$u=0, \quad \text{when } r=0,$$

$$u=av_0, \quad \text{when } r=a,$$

and

$$u=rf(r), \quad \text{when } t=0.$$

These are the same as the equations we obtained for the case of a rod of length a, whose ends are kept at temperatures zero and av_0, its initial temperature being $rf(r)$. The problem of the symmetrical distribution of heat in a sphere of radius a is thus mathematically the same as that of the flow of heat in a rod of length a.

The case of a hollow sphere can be discussed in the same way.

65. Sphere. Radiation at the Surface r=a into a Medium at Zero Temperature. Initial Temperature f(r).

In this case the equations for v are as follows:

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}\right), \quad (0<r<a) \quad \text{.................(1)}$$

$$\frac{\partial v}{\partial r}+hv=0, \quad \text{when } r=a, \quad \text{.........................(2)}$$

and

$$v=f(r), \quad \text{when } t=0. \quad \text{..............................(3)}$$

Putting $u=vr$, we have

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial r^2}, \quad (0<r<a) \quad \text{.................(4)}$$

$$u=0, \quad \text{when } r=0, \quad \text{.........................(5)}$$

$$\frac{\partial u}{\partial r}+\left(h-\frac{1}{a}\right)u=0, \quad \text{when } r=a, \quad \text{.........................(6)}$$

and

$$u=rf(r), \quad \text{when } t=0. \quad \text{.........................(7)}$$

The problem is thus reduced to that of the flow of heat in a rod, one end being kept at zero temperature, while at the other end radiation takes place into a medium at zero.

Proceeding as in § 36, we consider the expression

$$e^{-\kappa\alpha^2 t}\sin\alpha r.$$

This satisfies (4) and (5) whatever a may be, and it satisfies (6) if a is a root of the equation

$$aa \cos aa + (ah-1) \sin aa = 0. \qquad \text{.....................(8)}$$

To find the nature of the roots of this equation put

$$aa = \xi,$$

and we see that they correspond to the abscissae of the common points of the curves

$$\eta = \tan \xi \quad \text{and} \quad \eta = -\frac{\xi}{p},$$

where $p = ah - 1 > -1$.

The roots are thus symmetrically situated on the axis of ξ with regard to the origin. When $-1 < p < 0$, they lie in the intervals $(0, \frac{1}{2}\pi) : (\pi, \frac{3}{2}\pi) : \ldots$, and approach $\frac{1}{2}(2n-1)\pi$ as n increases; when $0 < p < \infty$, they lie in the intervals $(\frac{1}{2}\pi, \pi) : (\frac{3}{2}\pi, 2\pi) : \ldots$, and approach $\frac{1}{2}(2n-1)\pi$ as n increases; and there are no repeated roots.

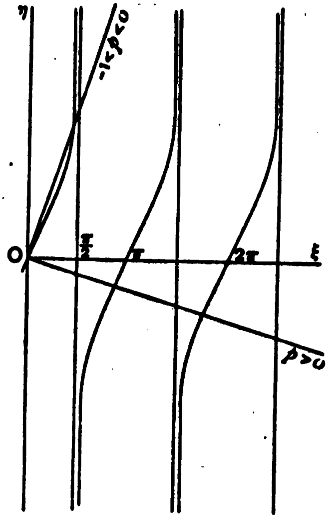

FIG. 8.

Further there can be no pure imaginary root of (8). For, if possible, let $aa = i\mu$, where μ is real.

Then $\quad F(\mu) = \mu + (ah-1) \tanh \mu = 0 \quad \text{and} \quad F(0) = 0.$

Also $\quad F'(\mu) = 1 + (ah-1) \operatorname{sech}^2 \mu.$

But $$ah - 1 > -1.$$

Therefore $F'(\mu)$ never vanishes, and $F(\mu)=0$ has no other real root than zero.

Also, there can be no imaginary roots of the form $\lambda \pm i\mu$.

For, if possible, let $a=\lambda+i\mu$ be such a root.

Then $a'=\lambda-i\mu$ must also be a root.

Then since
$$V=\sin ar,$$
$$V'=\sin a'r,$$
satisfy
$$\frac{d^2V}{dr^2}+a^2V=0 \quad \text{and} \quad \frac{d^2V'}{dr^2}+a'^2V'=0,$$
we have
$$(a^2-a'^2)\int_0^a VV'\,dr=\left[V\frac{dV'}{dr}-V'\frac{dV}{dr}\right]_0^a.$$

But V, V' satisfy (6), and therefore
$$\left[V\frac{dV'}{dr}-V'\frac{dV}{dr}\right]_0^a=0.$$

Thus
$$\int_0^a VV'\,dr=0.$$

Thus
$$\int_0^a \sin(\lambda+i\mu)r\sin(\lambda-i\mu)r\,dr=0,$$
and this requires that
$$\int_0^a(\sin^2\lambda r\cosh^2\mu r+\cos^2\lambda r\sinh^2\mu r)\,dr=0,$$
which is impossible.

We have thus shown that (8) has only real roots, and that these are infinite in number.

Let
$$a_1,\ a_2,\ \ldots$$
denote the positive roots in ascending order.

Then the series
$$\sum_1^\infty A_n e^{-\kappa a_n^2 t}\sin a_n r$$
satisfies (4), (5), and (6), and will satisfy (7) if
$$rf(r)=\sum_1^\infty A_n\sin a_n r.^*$$

Assuming the possibility of this expansion and that the series may be integrated term by term, we can obtain the values of these coefficients as in § 36.

* On the possibility of this expansion, see Ford, *loc. cit.*, p. 144.

We first show that

$$\int_0^a \sin \alpha_m r \sin \alpha_n r \, dr = 0 \quad (m \neq n)$$

and
$$\int_0^a \sin^2 \alpha_n r \, dr = \frac{a}{2} \frac{a^2\alpha_n^2 + ah(ah-1)}{a^2\alpha_n^2 + (ah-1)^2}.$$

These follow at once by integration. For

$$\int_0^a \sin \alpha_m r \sin \alpha_n r \, dr = \frac{\sin(\alpha_m - \alpha_n)a}{2(\alpha_m - \alpha_n)} - \frac{\sin(\alpha_m + \alpha_n)a}{2(\alpha_m + \alpha_n)}$$

$$= \frac{\cos \alpha_m a \cos \alpha_n a}{(\alpha_m^2 - \alpha_n^2)} (\alpha_n \tan \alpha_m a - \alpha_m \tan \alpha_n a).$$

and this vanishes, since α_m and α_n are roots of

$$\frac{\tan \alpha a}{\alpha} = \frac{a}{1 - ah}.$$

Also
$$\int_0^a \sin^2 \alpha_n r \, dr = \frac{1}{2}\int_0^a (1 - \cos 2\alpha_n r) dr$$

$$= \frac{a}{2} - \frac{\sin 2\alpha_n a}{4\alpha_n},$$

and
$$\sin 2\alpha_n a = \frac{2 \tan \alpha_n a}{1 + \tan^2 \alpha_n a}$$

$$= -\frac{2\alpha_n a(ah-1)}{(ah-1)^2 + a^2\alpha_n^2}.$$

Therefore
$$\int_0^a \sin^2 \alpha_n r \, dr = \frac{a}{2} \frac{a^2\alpha_n^2 + ah(ah-1)}{a^2\alpha_n^2 + (ah-1)^2}.$$

Hence, assuming that

$$rf(r) = A_1 \sin \alpha_1 r + A_2 \sin \alpha_2 r + \dots,$$

we have
$$A_n \int_0^a \sin^2 \alpha_n r \, dr = \int_0^a rf(r) \sin \alpha_n r \, dr$$

and
$$A_n = \frac{2}{a} \frac{a^2\alpha_n^2 + (ah-1)^2}{a^2\alpha_n^2 + ah(ah-1)} \int_0^a rf(r) \sin \alpha_n r \, dr.$$

Then the solution of our problem is

$u = vr$

$v = \frac{u}{r}$

$$u = \sum_1^\infty A_n e^{-\kappa\alpha_n^2 t} \sin \alpha_n r,$$

which gives

$$v = \frac{2}{ar} \sum_{n=1}^\infty \frac{a^2\alpha_n^2 + (ah-1)^2}{a^2\alpha_n^2 + ah(ah-1)} \left(\int_0^a r' f(r') \sin \alpha_n r' \, dr'\right) \sin \alpha_n r \, e^{-\kappa\alpha_n^2 t}.$$

This solution has been applied to the problem of Terrestrial Temperature, the initial temperature being constant, and the radius of the sphere very great. The solution of the problem of radiation in the Semi-Infinite Solid with arbitrary initial distribution of temperature may also be deduced from the result of this article.*

66. Application to the Determination of the Conductivities of Poor Conductors.

The expression we have just obtained for the temperature in a sphere cooling by radiation at the surface converges so rapidly that when a sufficient time has passed the terms after the first may be neglected. This gives an expression suitable for numerical calculation, and it has been applied in different experiments where the initial temperature of the sphere is constant.

For example, a ball of the material to be tested is immersed in a bath at a constant temperature V for a sufficient time to allow the whole ball to acquire the temperature of the bath. It is then removed and allowed to cool by radiation in a medium at constant temperature. After the cooling has gone on for a certain time, observations of the temperature are taken. In one set of experiments these readings are for the temperatures at the centre and the surface. In another set of experiments the temperature at the centre alone is required.

With the notation of § 65,

$$v=\frac{A_1}{r}e^{-\kappa a_1^2 t}\sin a_1 r \text{ to our approximation.}$$

Hence, if v_a = the temperature at $r=a$ at the time t

and v_0 = the temperature at $r=0$ at the time t,

$$\frac{v_a}{v_0}=\frac{\sin a_1 a}{a_1 a}. \qquad (1)$$

Also
$$\frac{(v_a)_{t=t_1}}{(v_a)_{t=t_2}}=e^{\kappa a_1^2(t_2-t_1)}. \qquad (2)$$

κa_1^2 is given by (2) and a_1 by (1), remembering that $0 < aa_1 < \pi$. Thus κ is obtained.

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* Cf. Riemann, *Partielle Differentialgleichungen*, §§ 69-70, Braunschweig, 1869; Weber-Riemann, *loc. cit.*, Bd. II., § 55.

Further, the original equation for α, namely

$$a\alpha \cos a\alpha + (ah-1) \sin a\alpha = 0,$$

gives the value of h.*

Ayrton and Perry used the second method in determining the Conductivity of Stone.† The temperature at the centre at time t is approximately

$$A_1 \alpha_1 e^{-\kappa\alpha_1^2 t}.$$

But
$$A_1 = v_0 \frac{\int_0^a r \sin \alpha_1 r \, dr}{\int_0^a \sin^2 \alpha_1 r \, dr}.$$

Therefore
$$A_1 = \frac{2v_0}{\alpha_1}\left(\frac{\sin \alpha_1 a - \alpha_1 a \cos \alpha_1 a}{\alpha_1 a - \sin \alpha_1 a \cos \alpha_1 a}\right).$$

Thus
$$\begin{aligned} v &= 2v_0 \frac{\sin \alpha_1 a - \alpha_1 a \cos \alpha_1 a}{\alpha_1 a - \sin \alpha_1 a \cos \alpha_1 a} e^{-\kappa\alpha_1^2 t} \\ &= Ne^{-nt}, \text{ say.} \end{aligned}$$

The value of n is obtained by two observations of the temperature, and n being known, the value of N may be found. Also a table of the values of the expression

$$\frac{\sin x - x \cos x}{x - \sin x \cos x}$$

will give α_1 from the known value of N.

But $n = \kappa\alpha_1^2$, and thus the conductivity is determined.

67. Sphere. Surface r=a at Zero Temperature. Initial Temperature f(r, θ, φ).

In this case the equations for v are as follows:

$$\frac{\partial v}{\partial t} = \kappa\left\{\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}\right\}. \quad \text{......(1)}$$

$$v = f(r, \theta, \phi), \text{ when } t = 0, \quad \text{......................(2)}$$

$$v = 0, \quad \text{when } r = a. \quad \text{......................(3)}$$

Put $v = e^{-\kappa\alpha^2 t}u$, where u is a function of r, θ and ϕ only, and write $\mu = \cos\theta$.

Then we have from (1),

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial v}{\partial\mu}\right) + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 u}{\partial\phi^2} + \alpha^2 u = 0. \quad \text{.....(4)}$$

* Weber, R., *Zurich, Vierteljahrsch. Natf. Ges.*, 23, p. 209, 1878.

† Cf. *Phil. Mag., London* (Ser. 5), 5, 1878.

Now the Zonal Harmonic $P_n(\mu)$, when n is a positive integer, is the coefficient of h^n in the expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$, and it satisfies Legendre's equation,

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_n}{d\mu}\right)+n(n+1)P_n=0.$$

Also $w=(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)$ satisfies

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dw}{d\mu}\right)+\left(n(n+1)-\frac{m^2}{1-\mu^2}\right)w=0,^*$$

where we have written D for $\dfrac{d}{d\mu}$.

It follows that the expression

$$R_n(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)\,\frac{\cos}{\sin}\,m\phi$$

will satisfy (4) provided that R_n is a function of r only, and

$$\frac{d^2R_n}{dr^2}+\frac{2}{r}\frac{dR_n}{dr}+\left(a^2-\frac{n(n+1)}{r^2}\right)R_n=0.$$

This leads us to $\qquad R_n=(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar),$

the solution $J_{-(n+\frac{1}{2})}(ar)$ being inadmissible, as it would make R_n tend to infinity as $r\to 0$.

We are thus brought to the following solution of (1):

$$v=e^{-\kappa a^2t}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)\,\frac{\cos}{\sin}\,m\phi, \quad \text{......(5)}$$

m and n being positive integers.

The condition at the surface is satisfied by (5), if a is a root of

$$J_{n+\frac{1}{2}}(aa)=0. \quad \text{..............................(6)}$$

If, as before, we assume that $f(r, \theta, \phi)$ can be expanded in a series whose terms are of the form

$$(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)\,\frac{\cos}{\sin}\,m\phi,$$

and that this series can be integrated term by term, we can find the coefficients in the expansion.

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* Byerly, *loc. cit.*, p. 196 (11), Boston, 1893; Whittaker and Watson, *Modern Analysis* (3rd Ed.), § 15. 5, 1920.

The second solution of this equation has an infinity at $\theta=\pi$, and is thus unsuitable for our problem.

For, let

$$f(r,\theta,\phi)=\sum_{a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)$$
$$\times\{A_{a,m,n}\cos m\phi+B_{a,m,n}\sin m\phi\},$$

the summation in a being over the positive roots of (6).

Then we have

$$\int_0^{2\pi}f(r,\theta,\phi)\cos m\phi\,d\phi$$
$$=\pi\sum_{a}\sum_{n=0}^{\infty}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)\,A_{a,m,n}.$$

Also we know* that

$$\left.\begin{aligned}&\int_{-1}^{1}(1-\mu^2)^m(D^mP_n(\mu))^2d\mu=\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\\ \text{and}\quad&\int_{-1}^{1}(1-\mu^2)^mD^mP_n(\mu)D^mP_{n'}(\mu)\,d\mu=0,\quad n\neq n'.\end{aligned}\right\}$$

Therefore

$$\int_{-1}^{1}(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)\,d\mu\int_0^{2\pi}f(r,\theta,\phi)\cos m\phi\,d\phi$$
$$=\frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\sum_a A_{a,m,n}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar).$$

Finally, from § 56, we obtain

$$\int_0^a r^{\frac{3}{2}}J_{n+\frac{1}{2}}(ar)\,dr\int_{-1}^{1}(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)\,d\mu\int_0^{2\pi}f(r,\theta,\phi)\cos m\phi\,d\phi$$
$$=A_{a,m,n}\frac{\pi a^2a^{-\frac{1}{2}}}{2n+1}\frac{(n+m)!}{(n-m)!}(J'_{n+\frac{1}{2}}(aa))^2.$$

In these results π must be replaced by 2π when $m=0$.

Also $B_{a,m,n}$ can be found in the same way.

Thus we are led to the solution of our problem in the form

$$v=\sum_a\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}e^{-\kappa a^2t}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)$$
$$\times\{A_{a,m,n}\cos m\phi+B_{a,m,n}\sin m\phi\},$$

the constants $A_{a,m,n}$ and $B_{a,m,n}$ being determined as above, the summation in a being over the positive roots of the equation $J_{n+\frac{1}{2}}(aa)=0.$

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* Cf. Byerly, *loc. cit.*, § 106; Whittaker and Watson, *loc. cit.* (3rd Ed.), § 15. 51.

It has been shown in § 9 that, when the surface temperature is not zero, the problem may be reduced to a case of steady temperature, and a case of variable temperature with zero surface temperature.

When the surface of the sphere is kept at $v=F(\theta, \phi)$, the steady temperature or potential problem, is given by

$$\nabla^2 u=0, \text{ through the sphere,}$$
$$u=F(\theta, \phi), \text{ at the surface,}$$

and we have*

$$u=\frac{1}{4\pi}\sum_0^\infty(2n+1)\left(\frac{r}{a}\right)^n\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} F(\theta', \phi') P_n(\cos\gamma)\, d\phi',$$

where the ordinary notation of Spherical Harmonics is employed.

The variable temperature problem will then be given by

$$\frac{\partial w}{\partial t}=\kappa\nabla^2 w,$$
$$w=f(r, \theta, \phi)-u, \text{ when } t=0,$$
$$w=0, \qquad \text{when } r=a.$$

And the solution of the problem with which we started is

$$v=u+w.$$

The corresponding questions, when radiation takes place at the surface, may be treated in the same way.†

68. The Part of the Sphere r = a cut out by the Cone $\theta=\theta_0$. Surface Temperature Zero. Initial Temperature f(r, θ, ϕ).

In this case the equations for v are as follows:

$$\frac{\partial v}{\partial t}=\kappa\left\{\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial v}{\partial\mu}\right)+\frac{1}{r^2(1-\mu^2)}\frac{\partial^2 v}{\partial\phi^2}\right\},$$

where $\mu=\cos\theta$, ...(1)

$v=0$, when $r=a$ and when $\theta=\theta_0$, ...(2)

$v=f(r, \theta, \phi)$, when $t=0$. ......(3)

Proceeding as in § 67, we are led to the following solution of (1):

$$v=e^{-\kappa\alpha^2 t}(\alpha r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\alpha r)\, P_n^{-m}(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi, \quad \text{......(4)}$$

where $P_n^{-m}(\mu)$ is the generalized Legendre's function ‡ given by the equation:

$$P_n^{-m}(\mu)=\frac{1}{\Gamma(1+m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}} F(-n, n+1, 1+m, \tfrac{1}{2}(1-\mu)).$$

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* Cf. Byerly, *loc. cit*, § 114. † Cf. Heine, *loc. cit.*, Bd. II., §§ 84, 85.

‡ The best treatment of the generalized Legendre's functions will be found in a paper by Barnes, *Q.J. Math.*, *London*, 39, p. 97, 1908.

It is obvious that $P_{-n-1}^{-m}(\mu)$ is the same as $P_n^{-m}(\mu)$.

Also, in this case,

$$\left.\begin{array}{ll} m \text{ is zero or a positive integer,} & \\ n \text{ is a root greater than } -\tfrac{1}{2} \text{ of} & P_n^{-m}(\mu_0)=0, \\ \text{and} \quad a \text{ is a positive root of} & J_{n+\frac{1}{2}}(aa)=0. \end{array}\right\}$$

The surface conditions (2) are satisfied by the value of v in (4), and there is no infinity within the solid.

If, as before, we assume that the function $f(r, \theta, \phi)$ can be expanded in a series whose terms are of the form

$$(ar)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(ar)\, P_n^{-m}(\mu) \frac{\cos}{\sin} m\phi,$$

and that this series can be integrated term by term, we can find the coefficients in the expansion.

For, let

$$f(r, \theta, \phi) = \sum_a \sum_{m=0}^{\infty} \sum_n (ar)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(ar) P_n^{-m}(\mu) \times \{A_{a,m,n} \cos m\phi + B_{a,m,n} \sin m\phi\},$$

the summation in n being over the roots greater than $-\frac{1}{2}$ of $P_n^{-m}(\mu_0)=0$, and that in a being over the positive roots of $J_{n+\frac{1}{2}}(aa)=0$.

Then we have

$$\int_0^{2\pi} f(r, \theta, \phi) \cos m\phi \, d\phi = \pi \sum_a \sum_n A_{a,m,n} (ar)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(ar)\, P_n^{-m}(\mu).$$

Also it is known that,* when m is any positive number and n, n' are two different roots greater than $-\frac{1}{2}$ of $P_n^{-m}(\mu_0)=0$,

$$\left.\begin{aligned} &\int_{\mu_0}^1 P_n^{-m}(\mu) P_{n'}^{-m}(\mu')\, d\mu = 0, \\ &\int_{\mu_0}^1 (P_n^{-m}(\mu))^2 d\mu = -\frac{(1-\mu_0^2)}{2n+1} \frac{d}{dn} P_n^{-m}(\mu_0) \frac{d}{d\mu_0} P_n^{-m}(\mu_0). \end{aligned}\right\}$$

* Let $\quad u = P_n^{-m}(\mu) \quad$ and $\quad u' = P_{n'}^{-m}(\mu)$.

Then we have

$$\left.\begin{aligned} &\frac{d}{d\mu}\left((1-\mu^2)\frac{du}{d\mu}\right) + \left(n(n+1) - \frac{m^2}{1-\mu^2}\right)u = 0, \\ &\frac{d}{d\mu}\left((1-\mu^2)\frac{du'}{d\mu}\right) + \left(n'(n'+1) - \frac{m^2}{1-\mu^2}\right)u' = 0. \end{aligned}\right\}$$

Therefore

$$(n'-n)(n'+n+1)\int_{\mu_0}^1 uu'\,d\mu$$
$$= \int_{\mu_0}^1 \left\{u' \frac{d}{d\mu}\left((1-\mu^2)\frac{du}{d\mu}\right) - u\frac{d}{d\mu}\left((1-\mu^2)\frac{du'}{d\mu}\right)\right\} d\mu$$
$$= \left[(1-\mu^2)\left\{u'\frac{du}{d\mu} - u\frac{du'}{d\mu}\right\}\right]_{\mu_0}^1.$$

[*Note continued*, p. 146.

Thus we have

$$\int_{\mu_0}^1 P_n^{-m}(\mu)\,d\mu \int_0^{2\pi} f(r,\theta,\phi)\cos m\phi\,d\phi$$

$$= -\frac{\pi(1-\mu_0^2)}{2n+1}\frac{d}{dn}P_n^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0)\sum_a A_{a,m,n}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar).$$

Finally, from § 56, we obtain

$$\int_0^a r^{\frac{3}{2}}J_{n+\frac{1}{2}}(ar)\,dr\int_{\mu_0}^1 P_n^{-m}(\mu)\,d\mu\int_0^{2\pi} f(r,\theta,\phi)\cos m\phi\,d\phi$$

$$= -A_{a,m,n}\frac{\pi a^2 a^{-\frac{1}{2}}(1-\mu_0^2)}{2(2n+1)}\frac{d}{dn}P_n^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0)\left(J'_{n+\frac{1}{2}}(aa)\right)^2.$$

In these results π must be replaced by 2π, when $m=0$.

Further $B_{a,m,n}$ follows in the same way.

Thus we are brought to the solution of our problem in the form

$$v=\sum_a\sum_{m=0}^{\infty}\sum_n e^{-\kappa a^2 t}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)P_n^{-m}(\mu)$$

$$\times(A_{a,m,n}\cos m\phi+B_{a,m,n}\sin m\phi),$$

the constants $A_{a,m,n}$, $B_{a,m,n}$ being determined as above.

If the solid consists of the part of the sphere $r=a$ cut out by the cone $\theta=\theta_0$ and the planes $\phi=0$, $\phi=\phi_0$, the surfaces being kept at zero temperature, we expand $f(r,\theta,\phi)$ in the series

$$\sum_a\sum_{m=1}^{\infty}\sum_n A_{a,m,n}(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)P_n^{-\frac{m\pi}{\phi_0}}(\mu)\sin\frac{m\pi}{\phi_0}\phi,$$

on the same lines as above.

It follows that, when n, n' are two different roots of $P_n^{-m}(\mu_0)=0$ greater than $-\frac{1}{2}$,

$$\int_{\mu_0}^1 P_n^{-m}(\mu)P_{n'}^{-m}(\mu)\,d\mu=0.$$

Also $$\int_{\mu_0}^1\left(P_n^{-m}(\mu)\right)^2 d\mu=-\frac{(1-\mu_0^2)}{2n+1}\operatorname*{Lt}_{n'\to n}\left(\frac{P_{n'}^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0)}{n'-n}\right)$$

$$=-\frac{(1-\mu_0^2)}{2n+1}\frac{d}{dn}P_n^{-m}(\mu_0)\frac{d}{d\mu_0}P_n^{-m}(\mu_0),$$

when $P_n^{-m}(\mu_0)=0.$

69. The Cone.

The result of § 68, when the solid is bounded by the sphere $r=a$ and the cone $\theta=\theta_0$, can be put as follows:

$$v=-\frac{2}{\pi a^2 r^{\frac{1}{2}}(1-\mu_0^2)}\sum_{\alpha}\sum_{m}\sum_{n}\frac{(2n+1)J_{n+\frac{1}{2}}(\alpha r)P_n^{-m}(\mu)\,e^{-\kappa\alpha^2 t}}{(J'_{n+\frac{1}{2}}(\alpha a))^2\dfrac{d}{dn}P_n^{-m}(\mu_0)\dfrac{d}{d\mu_0}P_n^{-m}(\mu_0)}$$

$$\times\int_0^a r'^{\frac{3}{2}}J_{n+\frac{1}{2}}(\alpha r')\,dr'\int_{\mu_0}^1(P_n^{-m}(\mu'))^2 d\mu'\int_0^{2\pi}f(r',\theta',\phi')\cos m(\phi-\phi')\,d\phi', \quad (1)$$

the summation in n being over the roots greater than $-\frac{1}{2}$ of $P_n^{-m}(\mu_0)=0$, that in α over the positive roots of $J_{n+\frac{1}{2}}(\alpha a)=0$, and m being zero or a positive integer. But when $m=0$, the result has to be divided by 2.

The temperature problem for the cone $\theta=\theta_0$, when the surface is kept at zero, and the initial temperature is $f(r,\theta,\phi)$, can be deduced from (1) by letting $a\to\infty$.

Now it is known* that, when a is very large,

$$J_{n+\frac{1}{2}}(\alpha a)=\sqrt{\left(\frac{2}{\pi\alpha a}\right)}\cos\left(\alpha a-(n+1)\frac{\pi}{2}\right) \text{ approximately,}$$

and thus

$$J'_{n+\frac{1}{2}}(\alpha a)=-\sqrt{\left(\frac{2}{\pi\alpha a}\right)}\sin\left(\alpha a-(n+1)\frac{\pi}{2}\right) \text{ approximately.}$$

If α, $\alpha+\Delta\alpha$ are consecutive roots of $J_{n+\frac{1}{2}}(\alpha a)=0$, when a is very large, we have the approximations

$$\left.\begin{aligned} a\,\Delta\alpha&=\pi,\\ a^2\left(J'_{n+\frac{1}{2}}(\alpha a)\right)^2&=\frac{2a}{\pi\alpha}=\frac{2}{\alpha\,\Delta\alpha}.\end{aligned}\right\}$$

Using these results, and remembering that the summation in α will become a definite integral, the solution of our temperature problem for the cone is as follows:

$$v=-\frac{1}{\pi r^{\frac{1}{2}}(1-\mu_0^2)}\sum_{m=0}^{\infty}\sum_{n}\frac{(2n+1)P_n^{-m}(\mu)}{\dfrac{d}{dn}P_n^{-m}(\mu_0)\dfrac{d}{d\mu_0}P_n^{-m}(\mu_0)}\int_0^{\infty}\alpha e^{-\kappa\alpha^2 t}J_{n+\frac{1}{2}}(\alpha r)\,d\alpha$$

$$\times\int_0^{\infty}r'^{\frac{3}{2}}J_{n+\frac{1}{2}}(\alpha r')\,dr'\int_{\mu_0}^1(P_n^{-m}(\mu'))^2 d\mu'\int_0^{2\pi}f(r',\theta',\phi')\cos m(\phi-\phi')\,d\phi',$$

* Cf. Watson, *loc. cit.*, § 7. 21 (i); Gray and Mathews, *loc. cit.*, p. 40.

the summation in n being taken over the roots greater than $-\frac{1}{2}$ of $P_n^{-m}(\mu_0)=0$.

In the term corresponding to $m=0$, as noted above, π must be replaced by 2π.

If the solid is the part of the cone $\theta=\theta_0$ cut off by the planes $\phi=0$ and $\phi=\phi_0$ the remark at the end of § 68 applies.

The general case when the surface is given by

$$\left.\begin{matrix} r=a \\ r=b \end{matrix}\right\}, \quad \left.\begin{matrix} \theta=\theta_0 \\ \theta=\theta_1 \end{matrix}\right\} \quad \text{and} \quad \left.\begin{matrix} \phi=0 \\ \phi=\phi_0 \end{matrix}\right\}$$

can be treated in the same way. The results are more complicated as Bessel's functions of both kinds, and the generalized Legendre's functions of both kinds, must now be taken into account.

CHAPTER IX

THE USE OF SOURCES AND SINKS IN CASES OF VARIABLE TEMPERATURE*

70. Instantaneous Point Source.

Suppose that a sphere of radius a at temperature V is placed at $t=0$ in an infinite solid of the same material at zero temperature and left to cool.

If $u=vr$, we know from § 64 that the temperature v at the time t is given by the equations:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial r^2}, \\ &u=Vr, \quad \text{when } t=0,\ 0<r\leqq a, \\ &u=0, \quad \text{when } t=0,\ r>a, \\ &u=0, \quad \text{when } r=0. \end{aligned}\right\}$$

The solution is thus known (§ 18) to be

$$v=\frac{V}{2r\sqrt{(\pi\kappa t)}}\int_0^a r'\left(e^{-\frac{(r-r')^2}{4\kappa t}}-e^{-\frac{(r+r')^2}{4\kappa t}}\right)dr'$$

$$=\frac{V}{2r\sqrt{(\pi\kappa t)}}\,e^{-\frac{r^2}{4\kappa t}}\int_0^a r'e^{-\frac{r'^2}{4\kappa t}}\left(e^{\frac{rr'}{2\kappa t}}-e^{-\frac{rr'}{2\kappa t}}\right)dr'.$$

Expanding the integrand in powers of r', and assuming that a is small, we obtain the approximate solution

$$v=\frac{Va^3}{6\pi^{\frac{1}{2}}\kappa^{\frac{3}{2}}t^{\frac{3}{2}}}\,e^{-\frac{r^2}{4\kappa t}}\left(1+\left(\frac{r^2}{\kappa t}-6\right)\frac{a^2}{40\kappa t}\right).$$

Let $Q=\frac{4}{3}\pi a^3V$.

*This method is due to Kelvin. Cf. "Compendium of the Fourier Mathematics for the Conduction of Heat in Solids," *Mathematical and Physical Papers*, Vol. II., p. 41. See also, Hobson, *London, Proc. Math. Soc.*, 19, p. 279, 1889; Rayleigh, *Phil. Mag., London* (Ser. 6), 22, p. 381, 1911.

Then this result can be written

$$v = \frac{Q}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{r^2}{4\kappa t}}\left[1+\left(\frac{r^2}{\kappa t}-6\right)\frac{a^2}{40\kappa t}\right].$$

Now let the radius of the sphere tend to zero, Q remaining constant, and we are led to the following solution of the equation of conduction:

$$v = \frac{Q}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{r^2}{4\kappa t}}, \qquad \text{.........................(1)}$$

where $r^2 = x^2+y^2+z^2$.

As $t \to 0$ this value of v tends to zero everywhere except at the origin, where it becomes infinite. Also if we integrate v through the infinite solid at any time ($t>0$) we obtain Q.

The distribution of temperature given by (1) is said to be due to an *Instantaneous Point Source of Strength* Q at the origin. For an Instantaneous Point Source of Strength Q placed at the point (x', y', z'), we have in the same way

$$v = \frac{Q}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}.$$

This is the fundamental solution of the equation of conduction in three dimensions. The quantity of heat concentrated at the source is $Q\rho c$.

We might have started with a cube whose edges are of length h, instead of the sphere of radius a. If this cube is placed in the infinite solid as above and left to cool, we would have (§ 16)

$$v = \frac{V}{(2\sqrt{(\pi\kappa t)})^3}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}\,dx'\,dy'\,dz',$$

the centre of the cube being at the origin and its edges parallel to the axes.

This leads to the approximate solution,

$$v = \frac{Q}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{x^2+y^2+z^2}{4\kappa t}}\left[1+\left(\frac{x^2+y^2+z^2}{\kappa t}-6\right)\frac{h^2}{96\kappa t}\right],$$

where $Q = h^3 V$.

Also letting $h \to 0$ we have

$$v = \frac{Q}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{x^2+y^2+z^2}{4\kappa t}},$$

as above.

For Q write $\phi(t')\,dt'$ in (1), change t into $(t-t')$ and integrate from 0 to t.

Then we have another solution of the equation of conduction, namely,

$$v=\frac{1}{(2\sqrt{(\pi\kappa)})^3}\int_0^t \phi(t')\frac{e^{-\frac{r^2}{4\kappa(t-t')}}}{(t-t')^{\frac{3}{2}}}dt'.$$

This distribution of temperature is said to be due to a *Continuous Point Source of Strength* $\phi(t)$ from $t=0$ onwards.

If $\phi(t)$ is constant and equal to q, we have

$$v=\frac{q}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}}\int_0^t \frac{e^{-\frac{r^2}{4\kappa(t-t')}}}{(t-t')^{\frac{3}{2}}}dt'$$

$$=\frac{q}{4\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}}\int_{1/\sqrt{t}}^{\infty} e^{-\frac{r^2}{4\kappa}\tau^2}d\tau, \text{ on putting } \frac{1}{\sqrt{(t-t')}}=\tau.$$

Let $t\to\infty$ and this reduces to $v=q/4\pi\kappa r$, a steady temperature distribution where a constant supply of heat is continually introduced at the origin and spreads outwards in the infinite solid.

71. Spherical Surface Source.

Again let a spherical shell $a<r<a+h$ at temperature V be placed at $t=0$ in an infinite solid of the same material at zero temperature and left to cool.

The temperature at the time t will be given by

$$v=\frac{V}{2r(\pi\kappa t)^{\frac{1}{2}}}\int_a^{a+h} r'\left(e^{-\frac{(r-r')^2}{4\kappa t}}-e^{-\frac{(r+r')^2}{4\kappa t}}\right)dr'.$$

Put $Q=\frac{4}{3}\pi((a+h)^3-a^3)V$, and let $h\to 0$, Q remaining constant.

Then we obtain another solution of the equation of conduction, namely,

$$v=\frac{Q}{8\pi ar(\pi\kappa t)^{\frac{1}{2}}}\left(e^{-\frac{(r-a)^2}{4\kappa t}}-e^{-\frac{(r+a)^2}{4\kappa t}}\right).$$

This is the temperature due to an *Instantaneous Spherical Surface Source of Strength* Q.*

The temperature due to a *Constant Spherical Surface Source of Strength* q will be obtained by writing $q\,dt'$ for Q, changing t into $(t-t')$, and integrating from 0 to t.

---

* This solution can also be obtained by distributing the Instantaneous Point Sources of § 70 uniformly over the surface of the sphere and evaluating the surface integral.

Then we have

$$v=\frac{q}{8ar\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}}\int_0^t\left(e^{-\frac{(r-a)^2}{4\kappa(t-t')}}-e^{-\frac{(r+a)^2}{4\kappa(t-t')}}\right)\frac{dt'}{(t-t')^{\frac{1}{2}}}$$

$$=\frac{q}{8ar\kappa\pi^{\frac{1}{2}}}\int_{1/\sqrt{t}}^{\infty}\left(e^{-m^2\tau^2}-e^{-n^2\tau^2}\right)\frac{d\tau}{\tau^2},$$

where $\tau=\frac{1}{2\sqrt{(\kappa(t-t'))}}$, $m^2=(r-a)^2$ and $n^2=(r+a)^2$.

For the steady temperature problem, let $t\to\infty$, and we obtain $v=q/4\pi\kappa r$, when $r>a$, and $v=q/4\pi\kappa a$, when $r<a$.

72. Instantaneous Line Source.

Let the whole of the xy plane be initially at zero temperature except a square with centre at the origin and edges of length h parallel to the axes, the initial temperature of this square being constant and equal to V.

This two-dimensional problem has for its solution (§ 47)

$$v=\frac{V}{4\pi\kappa t}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}\,dx'\,dy'.$$

Now let $h\to 0$, while Vh^2 remains constant and equal to Q. Then, in the limit,

$$v=\frac{Q}{4\pi\kappa t}e^{-\frac{x^2+y^2}{4\kappa t}}.$$

This is the temperature due to an *Instantaneous Line Source coinciding with the axis of z of strength Q.* When the source coincides with the parallel to the axis of z through $(x', y', 0)$, we have

$$v=\frac{Q}{4\pi\kappa t}e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}\quad\text{.............................(1)}$$

This is the fundamental solution for two-dimensional problems, and it is usually referred to as due to an Instantaneous Source of Strength Q at (x', y'). It will be noticed that the quantity of heat along the source per unit length is $Q\rho c$.

Using polar coordinates, (1) may be written

$$v=\frac{Q}{4\pi\kappa t}e^{-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{4\kappa t}},$$

and this reduces, with the aid of a well-known integral in Bessel's functions,* to

$$v=\frac{Q}{2\pi}\int_0^\infty a e^{-\kappa a^2 t} J_0(aR)\,da, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $$R^2=r^2+r'^2-2rr'\cos(\theta-\theta').$$

The solution for the *Instantaneous Cylindrical Surface Source of Strength Q* corresponding to the *Spherical Surface Source* of § 71 can be obtained by distributing Line Sources uniformly over the cylinder $r=a$.

In this way the temperature at the time t at a point distant r from the axis is given by

$$v=\frac{Q}{8\pi^2\kappa t}e^{-\frac{a^2+r^2}{4\kappa t}}\int_0^{2\pi} e^{\frac{ar}{2\kappa t}\cos\theta'}\,d\theta'$$

$$=\frac{Q}{4\pi\kappa t}e^{-\frac{a^2+r^2}{4\kappa t}}J_0\left[\frac{iar}{2\kappa t}\right].$$

73. Instantaneous Plane Source.

Again let the infinite solid, except the slice between the planes $x=\pm\frac{1}{2}h$, be initially at zero temperature, this portion being initially at the constant temperature V.

Then we have (§ 16)

$$v=\frac{V}{2\sqrt{(\pi\kappa t)}}\int_{-\frac{1}{2}h}^{\frac{1}{2}h} e^{-\frac{(x-x')^2}{4\kappa t}}\,dx'.$$

Now let $h\to 0$, while Vh remains constant and equal to Q.

Then, in the limit, $$v=\frac{Q}{2\sqrt{(\pi\kappa t)}}e^{-\frac{x^2}{4\kappa t}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This is the temperature due to an *Instantaneous Plane Source of Strength Q over the plane $x=0$*. If the source coincides with the plane $x=x'$, we have

$$v=\frac{Q}{2\sqrt{(\pi\kappa t)}}e^{-\frac{(x-x')^2}{4\kappa t}}.$$

This is the fundamental solution for linear flow. The quantity of heat per unit area concentrated on the plane is $Q\rho c$.

For the case of flow of heat along a rod it is usual to refer to this solution as due to a *Point Source* at the point x'.

The diagrams in Figs. 9, 10 illustrate graphically the distribution of temperature due to this Instantaneous Point Source. The dotted

* Cf. Gray and Mathews, *loc. cit.*, p. 77 (158).

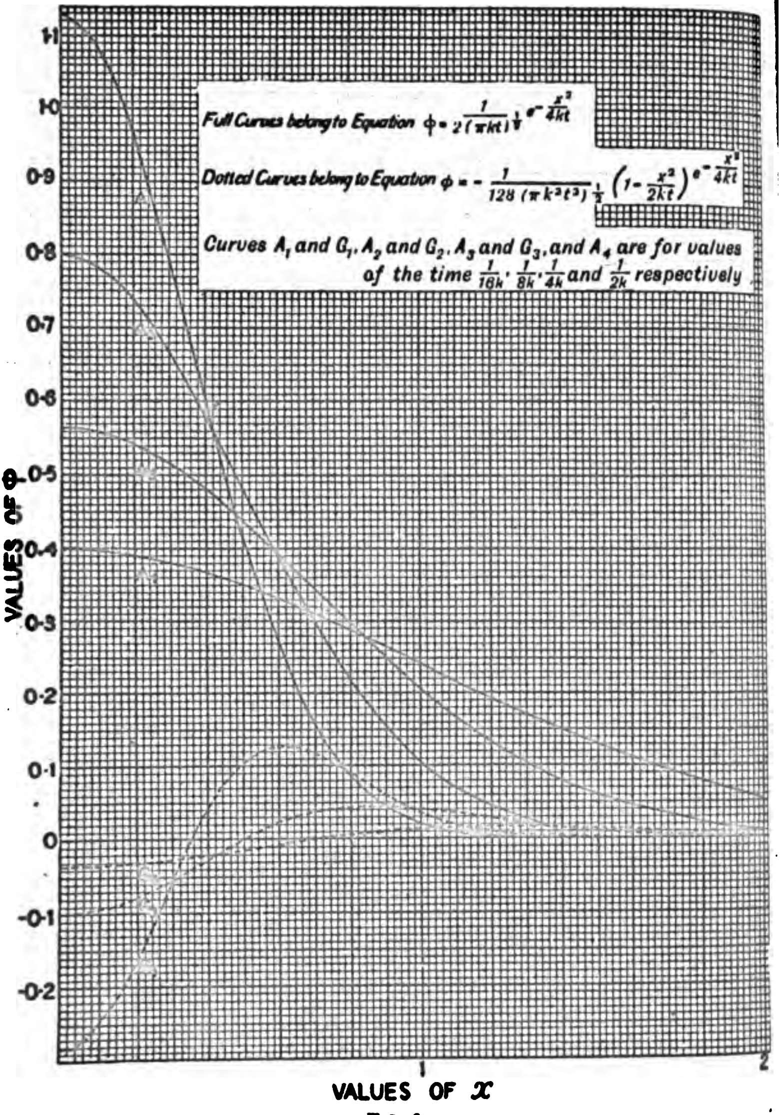

FIG. 9.

curve deals with $\frac{\partial v}{\partial t}$, since if $v = e^{-\frac{x^2}{4\kappa t}}/2(\pi\kappa t)^{\frac{1}{2}}$, it follows that

$$\frac{\partial v}{\partial t} = -\frac{1}{4(\pi\kappa t^3)^{\frac{1}{2}}}\left[1-\frac{x^2}{2\kappa t}\right]e^{-\frac{x^2}{4\kappa t}}.$$

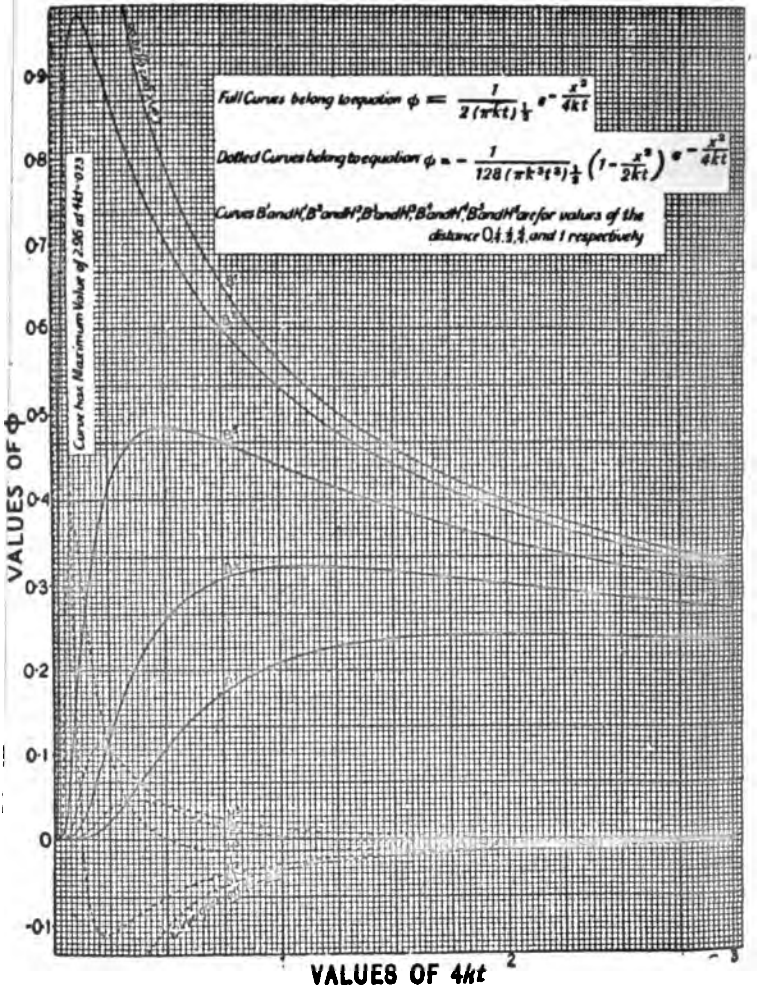

FIG. 10.

The temperature at the point $(x, y, 0)$ due to a uniform distribution of heat on the plane $x=0$ at the time $t=0$ could also be obtained by using the polar element of area in that plane.

In this way we have

$$v = \frac{Q}{(2\sqrt{(\pi\kappa t)})^3}\int_0^\infty \int_0^{2\pi} e^{-\frac{x^2+y^2+\rho^2-2y\rho\cos\theta}{4\kappa t}} \rho\, d\rho\, d\theta.$$

Thus
$$v = \frac{Q}{4\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}t^{\frac{3}{2}}} e^{-\frac{x^2+y^2}{4\kappa t}} \int_0^\infty \rho J_0\left(\frac{iy\rho}{2\kappa t}\right) e^{-\frac{\rho^2}{4\kappa t}} d\rho. \quad \text{.................(3)}$$

It follows from (1) and (2) that

$$\int_0^\infty \rho J_0\left(\frac{iy\rho}{2\kappa t}\right) e^{-\frac{\rho^2}{4\kappa t}} d\rho = 2\kappa t\, e^{\frac{y^2}{4\kappa t}}.$$

This is one of Weber's integrals in Bessel's functions, and a physical interpretation of the other integrals can be obtained by considering a more general distribution of heat along the plane $y=0$.*

74. Doublets.

We have seen that

$$v=\frac{Q}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}$$

is a solution of the equation of conduction

$$\frac{\partial v}{\partial t}=\kappa\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right].$$

It follows that

$$-\frac{\partial v}{\partial x}, \quad \text{or} \quad \frac{Q(x-x')}{2\kappa t(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}},$$

is also a solution.

This can be obtained by combining with a source of strength Q' at $(x'+dx', y', z')$ a sink† of strength $-Q'$ at (x', y', z'), letting $dx'\to 0$, and putting $\mathrm{Lt}\,(Q'dx')=Q$.

For the temperature at (x, y, z) due to the source and sink is given by

$$v=\frac{Q'}{(2\sqrt{(\pi\kappa t)})^3}\left[e^{-\frac{(x-x'-dx')^2+(y-y')^2+(z-z')^2}{4\kappa t}}-e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}\right].$$

Thus

$$v=\frac{Q'}{(2\sqrt{(\pi\kappa t)})^3} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}\left[e^{\frac{2(x-x')dx'-(dx')^2}{4\kappa t}}-1\right]$$

$$=\frac{Q'dx'}{16\pi^{\frac{3}{2}}\kappa^{\frac{5}{2}}t^{\frac{5}{2}}}(x-x')e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}+\text{higher powers of } dx'.$$

Proceeding to the limit, we obtain

$$v=\frac{Q(x-x')}{16\pi^{\frac{3}{2}}\kappa^{\frac{5}{2}}t^{\frac{5}{2}}} e^{-\frac{R^2}{4\kappa t}}, \quad \text{(1)}$$

where $R^2=(x-x')^2+(y-y')^2+(z-z')^2$.

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* Cf. Rayleigh, *loc. cit.*, p. 384; Gray and Mathews, *loc. cit.*, p. 78 (160).

† When the strength of a *source* as defined in § 70 is negative, it is called a *sink*.

The distribution of temperature in (1) is said to be due to an *instantaneous point doublet of strength Q at* (x', y', z'), *with its axis parallel to the axis of* x.

Corresponding definitions apply to the *doublet* in linear flow *at the point* x', and the *line doublet at* (x', y') *whose axis is parallel to the axis of* x in two-dimensional problems.

The temperature at the time t due to the two doublets just named will be

$$v = \frac{Q(x-x')}{4\pi^{\frac{1}{2}}\kappa^{\frac{3}{2}}t^{\frac{3}{2}}} e^{-\frac{(x-x')^2}{4\kappa t}}$$

and
$$v = \frac{Q(x-x')}{8\pi\kappa^2 t^2} e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}.$$

The extension to the *continuous doublet* of variable or constant strength is obvious. For example, the temperature due to a continuous doublet of strength $\phi(t)$ at the point x', in the case of linear flow, is given by

$$v = \frac{(x-x')}{4\pi^{\frac{1}{2}}\kappa^{\frac{3}{2}}} \int_0^t \phi(t') \frac{e^{-\frac{(x-x')^2}{4\kappa(t-t')}}}{(t-t')^{\frac{3}{2}}} dt'.$$

Substitute $x - x' = 2\kappa^{\frac{1}{2}}(t-t')^{\frac{1}{2}}a$, and we have

$$v = \frac{1}{\kappa\sqrt{\pi}} \int_{\frac{x-x'}{2\sqrt{(\kappa t)}}}^{\infty} e^{-a^2} \phi\left(t - \frac{(x-x')^2}{4\kappa a^2}\right) da, \text{ when } x > x',$$

and
$$v = \frac{1}{\kappa\sqrt{\pi}} \int_{\frac{x-x'}{2\sqrt{(\kappa t)}}}^{-\infty} e^{-a^2} \phi\left(t - \frac{(x-x')^2}{4\kappa a^2}\right) da, \text{ when } x < x'.$$

Thus
$$v_{x'+0} = \frac{\phi(t)}{\kappa\sqrt{\pi}} \int_0^{\infty} e^{-a^2} da = \frac{\phi(t)}{2\kappa}$$

and
$$v_{x'-0} = -\frac{\phi(t)}{2\kappa}.$$

Thus, in the case of linear flow in the semi-infinite solid $x > 0$, the plane $x=0$ can be kept at temperature $\phi(t)$ when $t > 0$, by placing a continuous doublet of strength $2\kappa\phi(t)$ at $x=0$. [Cf. § 23.]

In two-dimensional problems the boundary $y=0$ can be kept at temperature $f(x, t)$ by placing a continuous doublet at $(x', 0)$ with its axis parallel to the y-axis and strength $2\kappa f(x', t)\, dx'$, and integrating along the axis of x. A corresponding result holds for the three-

dimensional case when the plane $x=0$ is to be kept at temperature $f(y', z', t)$.*

75. The Method of Images.

The method of images, which plays so important a part in the Mathematical Theory of Electricity, is peculiarly adapted to the solution of problems in Conduction of Heat, when the solid is bounded by planes, and these are kept at zero temperature. We imagine the solid to be continued in all directions without limit, and we then obtain, by a suitable distribution of sources and sinks, a temperature function vanishing on the boundaries, with the required singularities (sources, sinks, etc.) in the solid. The distribution of sources and sinks outside the solid is in this case determined by taking images of the original distribution in the solid. We shall see in next chapter that the temperature due to a single source in the given solid, when its boundary is kept at zero, is of considerable importance in the solution of the general problem of conduction for that solid.

We proceed to discuss different types of problems whose solution can be obtained by the use of sources and sinks in this way.

Linear Flow.

I. *Semi-Infinite Solid* $x>0$. *Initial Temperature* $f(x)$. *Zero Temperature at the Boundary* $x=0$.

Consider the source of strength $f(x')dx'$ at the plane x'. We may take the initial temperature as due to a distribution of these sources along the positive axis of x.

With the source $f(x')dx'$ at x', we associate the sink $-f(x')dx'$ at $-x'$, as these two give zero temperature at $x=0$.

Hence $$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^\infty f(x')\left\{e^{-\frac{(x-x')^2}{4\kappa t}}-e^{-\frac{(x+x')^2}{4\kappa t}}\right\}dx'.$$

II. *Finite solid bounded by the planes* $x=0$ and $x=a$. *Initial Temperature* $f(x)$. *Bounding Planes kept at zero.*

Starting with the source $f(x')dx'$ at x', we have to take the images of this source in the planes $x=0$ and $x=a$, a source and a sink alternating so that the boundaries may be kept at zero. In this way we have sources at the points $x'+2na$ and sinks at the points $-x'+2na$, where n is zero or any positive or negative integer.

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* This statement can be verified from the results of §§ 86, 87.

Thus we have finally

$$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^a f(x')\left(\sum_{-\infty}^{\infty} e^{-\frac{(x-x'-2na)^2}{4\kappa t}}-\sum_{-\infty}^{\infty} e^{-\frac{(x+x'-2na)^2}{4\kappa t}}\right)dx'. \quad\text{......(1)}$$

We have already in § 30 obtained another expression for v in this case, namely,

$$\frac{2}{a}\sum_1^{\infty}\sin\frac{n\pi}{a}x\, e^{-\kappa\frac{n^2\pi^2}{a^2}t}\int_0^a f(x')\sin\frac{n\pi}{a}x'\,dx',$$

which may be written, when $t>0$,

$$\frac{2}{a}\int_0^a f(x')\sum_1^{\infty}\sin\frac{n\pi}{a}x\sin\frac{n\pi}{a}x'\,e^{-\kappa\frac{n^2\pi^2}{a^2}t}\,dx'. \quad\text{............(2)}$$

We proceed to show that these solutions are identical. This may be proved by the properties of the Theta-functions * or with the aid of the following theorem:

If $f(x)$ is an even function of x which can be expanded, as also $f(x\pm 2na)$, in a Fourier's Series of Cosines of multiples of $\pi x/a$, then

$$\sum_{-\infty}^{\infty} f(x+2na)=\frac{1}{a}\int_0^{\infty} f(x)\,dx+\frac{2}{a}\sum_1^{\infty}\cos\frac{n\pi}{a}x\int_0^{\infty} f(x')\cos\frac{n\pi}{a}x'\,dx',$$

provided the integrals are convergent and the series converges.

Since $$f(x)=\frac{1}{a}\int_0^a f(x')\,dx'+\frac{2}{a}\sum_1^{\infty}\cos\frac{n\pi}{a}x\int_0^a f(x')\cos\frac{n\pi}{a}x'\,dx',$$

and $$f(x+2na)=\frac{1}{a}\int_0^a f(x'+2na)\,dx'$$

$$+\frac{2}{a}\sum_1^{\infty}\cos\frac{n\pi}{a}x\int_0^a f(x'+2na)\cos\frac{n\pi}{a}x'\,dx'$$

$$=\frac{1}{a}\int_{2na}^{(2n+1)a} f(x')\,dx'+\frac{2}{a}\sum_1^{\infty}\cos\frac{n\pi}{a}x\int_{2na}^{(2n+1)a} f(x')\cos\frac{n\pi}{a}x'\,dx',$$

and $$f(x-2na)=\frac{1}{a}\int_{(2n-1)a}^{2na} f(x')\,dx'+\frac{2}{a}\sum_1^{\infty}\cos\frac{n\pi}{a}x\int_{(2n-1)a}^{2na} f(x')\cos\frac{n\pi}{a}x'\,dx',$$

it follows that

$$\sum_{-\infty}^{\infty} f(x+2na)=\frac{1}{a}\int_0^{\infty} f(x')\,dx'+\frac{2}{a}\sum_1^{\infty}\cos\frac{n\pi}{a}x\int_0^{\infty} f(x')\cos\frac{n\pi}{a}x'\,dx'.$$

* Cf. Poincaré, *Théorie de la propagation de la Chaleur*, p. 91; Whittaker and Watson, *loc. cit.* (3rd Ed.), p. 475.

Let $f(x)=e^{-\frac{x^2}{4\kappa t}}$, and we have *

$$\sum_{-\infty}^{\infty} e^{-\frac{(x+2na)^2}{4\kappa t}} = \frac{1}{a}\int_0^\infty e^{-\frac{x'^2}{4\kappa t}}\,dx' + \frac{2}{a}\sum_1^\infty \cos\frac{n\pi}{a}x\int_0^\infty e^{-\frac{x'^2}{4\kappa t}}\cos\frac{n\pi}{a}x'\,dx'$$

$$=\frac{\sqrt{(\pi\kappa t)}}{a}\left[1+2\sum_1^\infty \cos\frac{n\pi}{a}x\,e^{-\kappa\frac{n^2\pi^2}{a^2}t}\right].$$

Therefore

$$\sum_{-\infty}^{\infty} e^{-\frac{(x-x'+2na)^2}{4\kappa t}} = \frac{\sqrt{(\pi\kappa t)}}{a}\left[1+2\sum_1^\infty \cos\frac{n\pi}{a}(x-x')\,e^{-\kappa\frac{n^2\pi^2}{a^2}t}\right] \quad\text{.........(3)}$$

and

$$\sum_{-\infty}^{\infty} e^{-\frac{(x+x'+2na)^2}{4\kappa t}} = \frac{\sqrt{(\pi\kappa t)}}{a}\left[1+2\sum_1^\infty \cos\frac{n\pi}{a}(x+x')\,e^{-\kappa\frac{n^2\pi^2}{a^2}t}\right]. \quad\text{.........(4)}$$

Using (3) and (4) the solution (1) reduces to

$$v=\frac{2}{a}\int_0^a f(x')\sum_1^\infty e^{-\kappa\frac{n^2\pi^2}{a^2}t}\sin\frac{n\pi}{a}x\sin\frac{n\pi}{a}x'\,dx'.$$

III. *The Same Solid. Initial Temperature zero. Boundary $x=0$ kept at temperature $\phi_1(t)$. Boundary $x=a$ kept at zero.*

Starting with the Continuous Doublet of strength $2\kappa\phi_1(t)$ at $x=0$, which would keep $x=a$ at temperature $\phi_1(t)$ if the solid extended to infinity, we have to take an equal doublet at $x=2a$ to keep the plane $x=a$ at zero; and so on.

Thus we have doublets of strength $2\kappa\phi_1(t)$ at the points $2na$, n being zero, or any positive or negative integer.

Therefore

$$v=\frac{1}{2\sqrt{(\pi\kappa)}}\int_0^t \frac{\phi_1(t')}{(t-t')^{\frac{3}{2}}}\sum_{-\infty}^{\infty}\left((x+2na)\,e^{-\frac{(x+2na)^2}{4\kappa(t-t')}}\right)dt'.$$

A corresponding result may be obtained for the case when the boundary $x=0$ is kept at zero and $x=a$ at $\phi_2(t)$, and by addition of these solutions we are led to another form for the expression for the temperature in the problem of § 34.

76. Application of the Method of Images to Fourier's Ring.

I. *Ring of Unit Radius. Source of strength Q at $x=0$. Initial Temperature zero.*

Consider the problem of the rod $-\pi<x<\pi$, with a source of strength Q at $x=0$, and no flow of heat across its boundaries. The

* Cf. footnote p. 30.

solution is obtained by putting equal sources at the points $2n\pi$, n being any integer, positive or negative; and the temperature is given by

$$v = \frac{Q}{2\sqrt{(\pi\kappa t)}} \sum_{-\infty}^{\infty} e^{-\frac{(x+2n\pi)^2}{4\kappa t}}.$$

It is clear that all the conditions for the temperature in the ring are satisfied by this solution.

II. *Ring of Unit Radius. Initial Temperature* $v=f(x)$.

Since, as above, the temperature due to a source of strength Q at x' is

$$\frac{Q}{2\sqrt{(\pi\kappa t)}} \sum_{-\infty}^{\infty} e^{-\frac{(x-x'+2n\pi)^2}{4\kappa t}},$$

for the initial distribution $f(x)$ we have

$$v = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{-\pi}^{\pi} f(x') \sum_{-\infty}^{\infty} \left(e^{-\frac{(x-x'+2n\pi)^2}{4\kappa t}}\right) dx'.$$

Using § 75 (3), this reduces to

$$v = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{-\pi}^{\pi} f(x') \left\{\sqrt{\left[\frac{\kappa t}{\pi}\right]}\left(1+2\sum_{1}^{\infty} \cos n(x-x') e^{-\kappa n^2 t}\right)\right\} dx'$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') \left[1+2\sum_{1}^{\infty} \cos n(x-x') e^{-\kappa n^2 t}\right] dx',$$

as in § 12.*

77. Application of the Method of Images to the Infinite Solid. When $-\infty < x < 0$ the conductivity is K_2, and when $0 < x < \infty$ the conductivity is K_1. Source at $x'(x' > 0)$.

We require a solution of the equations:

$$\frac{\partial v_1}{\partial t} = \kappa_1 \frac{\partial^2 v_1}{\partial x^2}, \quad (x > 0) \qquad (1)$$

$$\frac{\partial v_2}{\partial t} = \kappa_2 \frac{\partial^2 v_2}{\partial x^2}, \quad (x < 0) \qquad (2)$$

$$v_1 = v_2, \text{ when } x = 0, \qquad (3)$$

$$K_1 \frac{\partial v_1}{\partial x} = K_2 \frac{\partial v_2}{\partial x}, \text{ when } x = 0, \qquad (4)$$

and $\underset{t=0}{\mathrm{Lt}}\, v_1 = 0$, except at $x = x'$, where v_1 is to take the form

$$\frac{1}{2\sqrt{(\pi\kappa_1 t)}} e^{-\frac{(x-x')^2}{4\kappa_1 t}}, \qquad (5)$$

and

$$\underset{t=0}{\mathrm{Lt}}\, v_2 = 0. \quad (x < 0) \qquad (6)$$

* Cf. Niven, *London, Proc. R. Soc.*, 76 (A), p. 41, 1905.

Differentiating (3) and (4) with respect to t and using (1) and (2), we have at $x=0$,

$$\kappa_1 \frac{\partial^2 v_1}{\partial x^2} = \kappa_2 \frac{\partial^2 v_2}{\partial x^2},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\kappa_1^n \frac{\partial^{2n} v_1}{\partial x^{2n}} = \kappa_2^n \frac{\partial^{2n} v_2}{\partial x_2^{2n}},$$

and
$$K_1\kappa_1 \frac{\partial^3 v_1}{\partial x^3} = K_2\kappa_2 \frac{\partial^3 v_2}{\partial x^3},$$

$$\dots\dots\dots\dots\dots\dots$$

$$K_1\kappa_1^n \frac{\partial^{2n+1} v_1}{\partial x^{2n+1}} = K_2\kappa_2^n \frac{\partial^{2n+1} v_2}{\partial x^{2n+1}}.$$

Now, using Maclaurin's Theorem,

$$v_2 = (v_2)_0 + x\left(\frac{\partial v_2}{\partial x}\right)_0 + \dots,$$

we have
$$2v_2 = (1+a)\sum_0^\infty \frac{1}{n!}\left(\sqrt{\left(\frac{\kappa_1}{\kappa_2}\right)}x\right)^n\left(\frac{\partial^n v_1}{\partial x^n}\right)_0$$
$$+(1-a)\sum_0^\infty \frac{1}{n!}\left(-\sqrt{\left(\frac{\kappa_1}{\kappa_2}\right)}x\right)^n\left(\frac{\partial^n v_1}{\partial x^n}\right)_0,$$

where
$$a = \frac{K_1\sqrt{\kappa_2}}{K_2\sqrt{\kappa_1}}.$$

This holds for any integral of (1), (2), (3) and (4).

Take first the expression $\quad v_1 = \dfrac{1}{\sqrt{t}} e^{-\frac{(x-x')^2}{4\kappa_1 t}}$; ....(7)

which satisfies (1).

Then we have

$$2v_2 = \frac{1+a}{\sqrt{t}} e^{-\frac{(\sqrt{(\kappa_1/\kappa_2)}x - x')^2}{4\kappa_1 t}} + \frac{1-a}{\sqrt{t}} e^{-\frac{(\sqrt{(\kappa_1/\kappa_2)}x + x')^2}{4\kappa_1 t}}$$

$$= \frac{1+a}{\sqrt{t}} e^{-\frac{(x-\sqrt{(\kappa_2/\kappa_1)}x')^2}{4\kappa_2 t}} + \frac{1-a}{\sqrt{t}} e^{-\frac{(x+\sqrt{(\kappa_2/\kappa_1)}x')^2}{4\kappa_2 t}}. \quad \dots\dots(8)$$

Now it is clear that this is not the solution we require, as the second term involves a source at the point $-\sqrt{(\kappa_2/\kappa_1)}\,x'$ in the part of the solid where $-\infty < x < 0$. We are able to obtain the true solution by considering what we would obtain from a sink at $-\sqrt{(\kappa_2/\kappa_1)}\,x'$, such as this second term requires, in the same way as above.

Exactly as in (8) we have, corresponding to the solution of (2),

$$v_2 = \frac{1}{\sqrt{t}} e^{-\frac{(x+\xi)^2}{4\kappa_2 t}},$$

$$2v_1 = \left(1+\frac{1}{a}\right)\frac{1}{\sqrt{t}} e^{-\frac{(x+\sqrt{(\kappa_1/\kappa_2)}\,\xi)^2}{4\kappa_1 t}} + \left(1-\frac{1}{a}\right)\frac{1}{\sqrt{t}} e^{-\frac{(x-\sqrt{(\kappa_1/\kappa_2)}\,\xi)^2}{4\kappa_1 t}}.$$

So that in the case of

$$v_2 = -\frac{(1-a)}{2\sqrt{t}} e^{-\frac{(x+\sqrt{(\kappa_2/\kappa_1)}x')^2}{4\kappa_2 t}}, \quad \dots\dots(9)$$

we have

$$v_1 = -\left(1+\frac{1}{a}\right)\left(\frac{1-a}{4\sqrt{t}}\right)e^{-\frac{(x+x')^2}{4\kappa_1 t}} - \left(1-\frac{1}{a}\right)\left(\frac{1-a}{4\sqrt{t}}\right)e^{-\frac{(x-x')^2}{4\kappa_1 t}} \quad \text{..........(10)}$$

Adding the solutions (7), (8), (9), and (10), we have

$$v_1 = \left(1+\frac{(a-1)^2}{4a}\right)\frac{e^{-\frac{(x-x')^2}{4\kappa_1 t}}}{\sqrt{t}} - \frac{1-a^2}{4a}\,\frac{e^{-\frac{(x+x')^2}{4\kappa_1 t}}}{\sqrt{t}},$$

and
$$v_2 = \frac{1+a}{2\sqrt{t}}\,e^{-\frac{(x-\sqrt{(\kappa_2/\kappa_1)}\,x')^2}{4\kappa_2 t}},$$

which satisfy all the conditions of the problem except for a numerical factor in the strength of the source at x'.

We obtain the actual solution by dividing both by

$$\frac{(a+1)^2\sqrt{(\pi\kappa_1)}}{2a},$$

and thus have

$$v_1 = \frac{1}{2\sqrt{(\pi\kappa_1 t)}}\,e^{-\frac{(x-x')^2}{4\kappa_1 t}} + \frac{a-1}{a+1}\,\frac{e^{-\frac{(x+x')^2}{4\kappa_1 t}}}{2\sqrt{(\pi\kappa_1 t)}}, \quad (x>0)$$

and
$$v_2 = \frac{2a}{1+a}\,\frac{e^{-\frac{(x-\sqrt{(\kappa_2/\kappa_1)}\,x')^2}{4\kappa_2 t}}}{2\sqrt{(\pi\kappa_1 t)}}. \qquad (x<0)$$

Thus the temperature at any time t in the part $x>0$, is the same as if the whole solid had been of that material and another source of strength

$$\frac{K_1\sqrt{\kappa_2}-K_2\sqrt{\kappa_1}}{K_1\sqrt{\kappa_2}+K_2\sqrt{\kappa_1}}$$

had been placed at $-x'$: and the temperature in the part $x<0$ is the same as if the whole solid had been of that material and a source of strength

$$\frac{2K_1\kappa_2}{(K_1\sqrt{\kappa_2}+K_2\sqrt{\kappa_1})},$$

had been placed at $\sqrt{(\kappa_2/\kappa_1)}\,x'$.*

78. Applications of the Method of Images in Two or Three Dimensions.

I. *Semi-Infinite Solid,* $x>0$. *Initial Temperature* $f(x, y)$. *Boundary* $x=0$ *kept at zero.*

Starting with the line source of strength $f(x', y')dx'\,dy'$ at (x', y'), we must take an equal sink at $(-x', y')$ to satisfy the condition at the boundary.

Hence

$$v = \frac{1}{4\pi\kappa t}\int_0^\infty dx'\int_{-\infty}^{\infty} f(x', y')\left\{e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}} - e^{-\frac{(x+x')^2+(y-y')^2}{4\kappa t}}\right\}dy'.$$

* Cf. Sommerfeld, *Math. Ann., Leipzig,* 45, p. 266, 1894; Weber, *Göttingen, Nachr. Ges. Wiss.,* p. 722, 1893; Weber-Riemann, *loc. cit.,* Bd. II., § 40.

II. *Semi-Infinite Solid, $x>0$. Initial Temperature $f(x, y, z)$. Boundary $x=0$ kept at zero.*

Starting with the point source of strength $f(x', y', z')dx'\,dy'\,dz'$ at (x', y', z'), we take an equal sink at $(-x', y', z')$, since these give zero temperature at $x=0$.

Hence

$$v=\frac{1}{(2\sqrt{(\pi\kappa t)})^3}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y', z')\left\{e^{-\frac{R^2}{4\kappa t}}-e^{-\frac{R'^2}{4\kappa t}}\right\}dx'\,dy'\,dz',$$

where $$R^2=(x-x')^2+(y-y')^2+(z-z')^2$$

and $$R'^2=(x+x')^2+(y-y')^2+(z-z')^2.$$

III. *The Wedge of Angle π/m, where m is any positive integer.*

The two and three-dimensional problems given above in I. and II. are special cases of the wedge of angle π/m, where m is any positive integer. We shall now treat this problem, confining ourselves to the two-dimensional case of a line source at the point (x', y'), the edge of the wedge coinciding with the axis of z. The three-dimensional case of a point source at (x', y', z'), and the extension to the general problem of an arbitrary initial temperature offer no difficulty.

Taking cylindrical coordinates, the surface of the wedge is supposed given by the planes $\theta=0$ and $\theta=\pi/m$; these planes are to be kept at zero temperature.

Within the wedge we have $0<\theta<\pi/m$.

Let the source be placed at the point P_0 whose coordinates are (a, α).

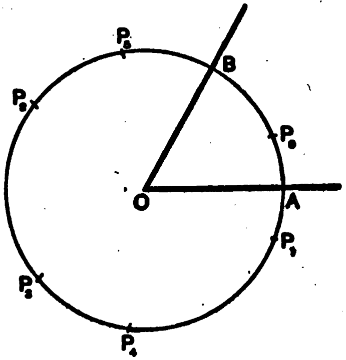

FIG. 11.

Let the circle through P_0 with its centre at the origin cut $\theta=0$ and $\theta=\pi/m$ at A and B (Fig. 11).

Then the angles AOP, POB and AOB are α, β and γ, where $\beta=\pi/m-\alpha$ and $\gamma=\pi/m$.

Start with the unit source at P_0.

To give zero temperature at OA we put a unit *sink* at P_1, the image of P_0 in OA: *i.e.* at $(-\alpha)$.

To balance the sink at P_1, in OB, we put a *source* at P_2, the image of P_1 in OB: *i.e.* at $(\alpha+2\gamma)$.

To balance the source at P_2 in OA, we put a *sink* at P_3, the image of P_2 in OA: *i.e.* at $-(\alpha+2\gamma)$: and so on.

In this way we have the set of images P_1, P_2, ... , where

$$P_0P_2=P_0P_4=\ldots=2\gamma,$$
$$P_1P_3=P_0P_5=\ldots=2\gamma.$$

Also P_{2m-1} lies at $-(\alpha+2(m-1)\gamma)$.

Thus $$P_0P_{2m-1}+2\beta=2\alpha+2(m-1)\gamma+2\beta=2\pi.$$

Therefore P_{2m-1} coincides with the image of P in OB, and the set of images is closed, P_{2m-1} being the last.

Also these sources and sinks give, with the source at P_0, zero temperature over the planes $\theta=0$ and $\theta=\pi/m$.

The temperature at (r, θ) due to this system is

$$v=\sum_{s=0}^{2m-1}(-1)^s v_s, \qquad\qquad (1)$$

where v_s is the temperature due to a unit source at P_s in the infinite solid.

But we have seen in § 72 (2) that the temperature at (r, θ) due to a unit source at (r', θ') is

$$\frac{1}{2\pi}\int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_0(\lambda R)\, d\lambda,$$

where $$R^2=r^2+r'^2-2rr'\cos(\theta-\theta').$$

Using Neumann's expansion *

$$J_0(\lambda R)=J_0(\lambda r)\, J_0(\lambda r')+2\sum_1^\infty J_n(\lambda r)\, J_n(\lambda r')\cos n(\theta-\theta'),$$

this may be written as

$$\frac{1}{2\pi}\int_0^\infty \lambda e^{-\kappa\lambda^2 t}\sum_{-\infty}^\infty J_n(\lambda r)\, J_n(\lambda r')\cos n(\theta-\theta')\, d\lambda,$$

or $$\frac{1}{2\pi}\sum_{-\infty}^\infty \cos n(\theta-\theta')\int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_n(\lambda r)\, J_n(\lambda r')\, d\lambda.$$

---

* Cf. Gray and Mathews, *loc. cit.*, p. 27 (69').

It follows from (1) that

$$v=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{s=0}^{m-1}(\cos n(\theta-a-2s\gamma)-\cos n(\theta+a+2s\gamma))$$
$$\times\int_0^\infty \lambda e^{-\kappa\lambda^2 t}J_n(\lambda r)\,J_n(\lambda a)\,d\lambda.\;\ldots(2)$$

When n is not a multiple of m the series in s has zero for its sum; and when it is a multiple of m, its sum is equal to

$$2m\sin n\theta\sin na.$$

Thus, from (2), we obtain the solution of our problem of the unit source at the point (a, a) in the wedge $\theta=0$, $\theta=\pi/m$, in the form

$$v=\frac{2}{\gamma}\sum_{p=1}^{\infty}\sin\frac{p\pi}{\gamma}\theta\sin\frac{p\pi}{\gamma}a\int_0^\infty\lambda e^{-\kappa\lambda^2 t}J_{\frac{p\pi}{\gamma}}(\lambda r)\,J_{\frac{p\pi}{\gamma}}(\lambda a)\,d\lambda,\;\ldots\ldots(3)$$

where, as above, we have written γ for π/m.

For the three-dimensional case, we start with the expression

$$\frac{1}{(2\sqrt{(\pi\kappa t)})^3}e^{-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')+(z-z')^2}{4\kappa t}},$$

corresponding to the unit source at (r', θ', z').

Proceeding on the same lines as above, we obtain the solution of our problem in the form

$$v=\frac{e^{-\frac{(z-z')^2}{4\kappa t}}}{\gamma\sqrt{(\pi\kappa t)}}\sum_{p=1}^{\infty}\sin\frac{p\pi}{\gamma}\theta\sin\frac{p\pi}{\gamma}a\int_0^\infty\lambda e^{-\kappa\lambda^2 t}J_{\frac{p\pi}{\gamma}}(\lambda r)\,J_{\frac{p\pi}{\gamma}}(\lambda r')\,d\lambda.$$

79. Sommerfeld's Extension of the Method of Images.

The method of images as used above for the wedge of angle π/m, where m is any positive integer, fails when the angle is $n\pi/m$, where m, n are both positive integers, prime to each other.

For example, when the angle is a right angle, and the given source is at $P_0(r', \theta')$, where $0<\theta<\frac{1}{2}\pi$, the images are as follows:

a sink at $P_1(r', -\theta)$: a source at $P_2(r', \pi+\theta')$;

and a sink at $P_3(r', -\pi-\theta')$. (Cf. Fig. 11.)

But when the angle is $2\pi/3$ and the given source is at

$P_0(r', \theta')$, where $0 < \theta' < 2\pi/3$, the successive images are as follows:

$$\text{a sink at } P_1(r', -\theta') : \text{ a source at } P_2\left(r', \frac{4\pi}{3}+\theta'\right);$$

$$\text{a sink at } P_3\left(r', -\frac{4\pi}{3}-\theta'\right) : \text{ a source at } P_4\left(r', \frac{8\pi}{3}+\theta'\right);$$

$$\text{and} \quad \text{a sink at } P_5\left(r', -\frac{8\pi}{3}-\theta'\right). \quad \text{(Fig. 12.)}$$

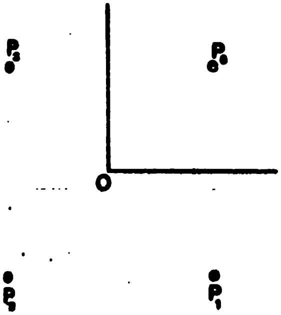

FIG. 12.

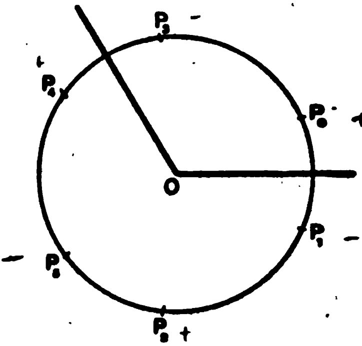

FIG. 12.

As the expression for the temperature due to a source is periodic of period 2π, and the sink at P_3 gives a singularity corresponding to a sink at $\left(r', \frac{2\pi}{3}-\theta'\right)$, the method fails.

However, for the complete representation of the solid we need only the region $0 < \theta < 2\pi/3$, and if we can find a solution of the equation of conduction which has a period 4π and only one singularity in that interval, and this of the proper kind, viz.

$$\frac{e^{-\frac{r^2}{4\kappa t}}}{4\pi\kappa t} \quad \text{or} \quad \frac{e^{-\frac{r^2}{4\kappa t}}}{(2\sqrt{(\pi\kappa t)})^3}, \text{ when } r=0 \text{ and } t=0,$$

we can use this solution as we did the ordinary expression for the temperature of a source and take the images at the points named above.

This method, as originally introduced by Sommerfeld, really amounts to considering a solution of the equation of conduction on a suitable Riemann's Surface (or Space). For the angle $n\pi/m$, the Riemann's Surface (or Space) will be an n-fold one, and the solution will have a period $2n\pi$. The method is of historical interest because,

after applying it to the heat problem of a source in the region bounded by the planes $\theta=0$ and $\theta=2\pi$, Sommerfeld by its aid gave the first exact solution of the diffraction of waves by a semi-infinite plane (*e.g.* $\theta=0$). But a simpler method of treating these questions both from the equation of conduction of heat and for the other partial differential equations of mathematical physics has now been evolved.* For this reason it will be sufficient here only to give references to Sommerfeld's and other papers in which the Riemann's Surface idea is used.†

We return to the problem of the wedge in § 90 and the solution obtained in § 78, III. (3) for the angle π/m will be found to be true for the wedge of any angle.

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* See below § 90 and papers by the author in *London, Proc. Math. Soc.* (Ser. 2), 8, 1910 and, 18, 1920.

† Sommerfeld, (1) *Math. Ann., Leipzig*, 45, p. 274, 1894, and 47, p. 317, 1896; (2) *London, Proc. Math. Soc.*, 28, p. 395, 1897. Schwarzschild, *Math. Ann., Leipzig*, 55, p. 177, 1892. Carslaw, (1) *London, Proc. Math. Soc.*, 30, p. 121, 1899; (2) *London, Rep. Brit. Ass.*, p. 644, 1900. Hobson, *Cambridge, Trans. Phil. Soc.*, 18, p. 277, 1900.

CHAPTER X

THE USE OF GREEN'S FUNCTIONS IN THE SOLUTION OF THE EQUATION OF CONDUCTION

80. Introductory.

The use of Green's functions in the theory of potential is well known. The function is most conveniently defined for the closed surface S as the potential which vanishes over the surface, and is infinite as $1/r$, when r is zero, at the point $P(x', y', z')$ inside the surface. If this solution of the equation $\nabla^2 u = 0$ is denoted by $G(P)$, the solution with no infinity inside S and an arbitrary value V over the surface is given by

$$u = \frac{1}{4\pi}\iint \frac{\partial}{\partial n} G(P)\, V\, dS,$$

$\frac{\partial}{\partial n}$ denoting differentiation along the outward drawn normal.*

We proceed to show how a similar function may be employed with advantage in the mathematical theory of the conduction of heat. In this case we shall take *the Green's function as the temperature at (x, y, z) at the time t due to an instantaneous point source of strength unity generated at the point $P(x', y', z')$ at the time τ, the solid being initially at zero temperature, and the surface being kept at zero temperature.*

This solution may be written

$$u = F(x, y, z, x', y', z', t-\tau), \quad (t > \tau)$$

and u satisfies the equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u. \quad (t > \tau)$$

* Cf. Clerk-Maxwell, *Electricity and Magnetism*, Vol. I., § 97 (*b*), and Webster, *Electricity and Magnetism*, p. 290.

However, since t only enters in the form $(t-\tau)$, we have also

$$\frac{\partial u}{\partial \tau}+\kappa\nabla^2 u=0. \quad (\tau<t)$$

Further, $\underset{t\to\tau}{\text{Lt}}(u)=0$ at all points inside S, except at the point (x', y', z'), where the solution takes the form

$$\frac{1}{\{2\sqrt{(\pi\kappa(t-\tau))}\}^3}e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}.$$

Finally, at the surface S, $u=0$. $(\tau<t.)$

Let v be the temperature at the time t in this solid due to the surface temperature $\phi(x, y, z, t)$ and the initial temperature $f(x, y, z)$.

Then v satisfies the equations

$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v, \quad (t>0)$$

$$v=f(x, y, z) \text{ initially, inside } S,$$

and $$v=\phi(x, y, z, t) \text{ at } S, \text{ when } t>0.$$

Also, since the time τ of our former equations lies within the interval for t, we have

$$\frac{\partial v}{\partial \tau}=\kappa\nabla^2 v, \quad (\tau<t)$$

$$v=\phi(x, y, z, \tau) \text{ at the surface.}$$

Therefore $$\frac{\partial}{\partial\tau}(uv)=u\frac{\partial v}{\partial\tau}+v\frac{\partial u}{\partial\tau}=\kappa[u\nabla^2 v-v\nabla^2 u],$$

and

$$\int_0^{t-\epsilon}\left[\iiint\frac{\partial}{\partial\tau}(uv)\,dx\,dy\,dz\right]d\tau=\kappa\int_0^{t-\epsilon}\left[\iiint(u\nabla^2 v-v\nabla^2 u)\,dx\,dy\,dz\right]d\tau,$$

the triple integration being taken throughout the solid, and ϵ being any positive number less than t, as small as we please.

Interchanging the order of integration on the left-hand side of this equation and applying Green's Theorem to the right-hand side, we have

$$\iiint(uv)_{\tau=t-\epsilon}\,dx\,dy\,dz-\iiint(uv)_{\tau=0}\,dx\,dy\,dz$$

$$=\kappa\int_0^{t-\epsilon}\left[\iint\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS\right]d\tau$$

$$=\kappa\int_0^{t-\epsilon}\left[\iint v\left(\frac{\partial u}{\partial n}\right)_i dS\right]d\tau,$$

where $\dfrac{\partial}{\partial n_i}$ denotes differentiation along the inward-drawn normal, and we have used the condition that u vanishes at the surface.

Now take the limit as ϵ tends to zero. The left-hand side gives

$$[v_P]_t\left[\iiint u_{\tau=t-0}\,dx\,dy\,dz\right]-\iiint(u)_{\tau=0}(v)_{\tau=0}\,dx\,dy\,dz,$$

the first integral being taken through an element of volume including the point $P(x', y', z')$, where the function u becomes infinite at $t=\tau$, the second integral being taken through the solid; and $[v_P]_t$ stands for the value of v at the point $P(x', y', z')$ at the time t.

But since u is the temperature at the time t due to a source at (x', y', z') at the time τ,

$$\iiint(u)_{\tau=t-0}\,dx\,dy\,dz=1,$$

and we have

$$[v_P]_t=\iiint(u)_{\tau=0}(v)_{\tau=0}\,dx\,dy\,dz+\kappa\int_0^t\left[\iint v\frac{\partial u}{\partial n_i}\,dS\right]d\tau$$

$$=\iiint(u)_{\tau=0}f(x, y, z)\,dx\,dy\,dz+\kappa\int_0^t\left[\iint\phi(x, y, z, \tau)\frac{\partial u}{\partial n_i}dS\right]d\tau \quad (1)$$

as the temperature at x', y', z' at the time t due to the initial distribution $f(x, y, z)$ and the surface temperature $\phi(x, y, z, t)$.*

In the case of Radiation at the surface, the Green's function u *is taken as the temperature at* (x, y, z) *at time* t *due to an instantaneous point source of strength unit generated at* (x', y', z') *at time* τ, *radiation taking place at the surface into a medium at zero temperature.*

The temperature at $P(x', y', z')$ at the time t due to an initial distribution $f(x, y, z)$ and radiation at the surface into a medium

* This discussion is due to Minnigerode, and was published in his Göttingen Dissertation, *Über die Wärmeleitung in Krystallen*, Göttingen, 1862. Cf. also Betti, (1) *Ann. delle Università Toscane*, 10, p. 143, 1868, Pisa; (2) *Ann. Mat., Milano*, 1, p. 373, 1868; (3) *Mem. Soc. Italiana delle Scienze* (Ser. 3), 1, p. 373, 1868, Firenze; (4) *Collectanea Mathematica inedita in Memoriam Domenici Chelini*, p. 238, 1881, Milano. Sommerfeld, *Math. Ann., Leipzig*, 45, p. 274, 1894. Weber-Riemann, *loc. cit.*, Bd. II., § 52, and papers by the author, (1) *Phil. Mag., London* (Ser. 6), 4, p. 162, 1902; (2) *Edinburgh, Proc. Math. Soc.*, 21, p. 40, 1903; (3) *London, Proc. Math. Soc.* (Ser. 2), 8, p. 365, 1910. See also for application to the equation $(\nabla^2+\kappa^2)u=0$, Pockels, *Über die Partielle-Differentialgleichung* $(\nabla^2+\kappa^2)u=0$, Tl. IV. § 4, Leipzig, 1891; Schwarzschild, *Math. Ann., Leipzig*, 55, p. 177, 1902.

at temperature $\phi(x, y, z, t)$ follows from a discussion similar to that given above. We find in the end

$$[v_P]_t = \iiint (u)_{\tau=0} f(x, y, z)\, dx\, dy\, dz + h\kappa \int_0^t \left[\iint u\phi(x, y, z, \tau)\, dS\right] d\tau$$

$$= \iiint (u)_{\tau=0} f(x, y, z)\, dx dy dz + \kappa \int_0^t \left[\iint \left(\frac{\partial u}{\partial n}\right)_i \phi(x, y, z, \tau)\, dS\right] d\tau, \quad (2)$$

since at the surface $\qquad \dfrac{\partial u}{\partial n_i} = hu,$

and our result takes the same form as in (1).

The solution of the general problems in conduction of heat is thus reduced to the determination of the Green's function for the solid in which the temperature is required.

In the case of linear or two-dimensional flow of heat results similar to (1) and (2) can be obtained at once. Instead of an infinity of order

$$\frac{1}{\{2\sqrt{(\pi\kappa t)}\}^3} e^{-\frac{R^2}{4\kappa t}},$$

we have $\qquad \dfrac{1}{2\sqrt{(\pi\kappa t)}} e^{-\frac{R^2}{4\kappa t}}$ and $\dfrac{1}{4\pi\kappa t} e^{-\frac{R^2}{4\kappa t}},$

respectively. With this change the equations which correspond to (1) and (2) will be

$$[v_P]_t = \int (u)_{\tau=0} f(x)\, dx + \kappa \int_0^t \phi(\tau) \frac{\partial u}{\partial n_i} d\tau, \quad \text{.......................}(3)$$

and $\qquad [v_P]_t = \iint (u)_{\tau=0} f(x, y)\, dx\, dy + \kappa \int_0^t \left[\int \phi(x, y, \tau) \dfrac{\partial u}{\partial n_i} ds\right] d\tau, \quad (4)$

where the integration with regard to s is along the bounding arc.

In the articles which follow we shall employ these theorems in solving various problems in the conduction of heat. Some of them have already been discussed by other methods, the possibility of the expansion of the arbitrary function in the form required in the solution being assumed. This assumption will not now be necessary.

81. Linear Flow of Heat. Semi-Infinite Solid bounded by x=0. Initial Temperature f(x). Boundary kept at Temperature ϕ(t).

In this case the Green's function, or the temperature at (x, y, z) at the time t due to the unit source at (x', y', z') at the time τ, is

$$\frac{1}{2\sqrt{(\pi\kappa(t-\tau))}} \left\{ e^{-\frac{(x-x')^2}{4\kappa(t-\tau)}} - e^{-\frac{(x+x')^2}{4\kappa(t-\tau)}} \right\}. \qquad = G$$

It follows from § 80 (3), with a slight change in the notation, that the temperature at (x, y, z) at the time t is given by

$$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^\infty f(x')\left\{e^{-\frac{(x-x')^2}{4\kappa t}}-e^{-\frac{(x+x')^2}{4\kappa t}}\right\}dx'$$

$$+\frac{x}{2\sqrt{(\pi\kappa)}}\int_0^t \phi(\tau)\frac{e^{-\frac{x^2}{4\kappa(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}d\tau,$$

which agrees with § 23.

82. The Same Solid. Source at x′ at t=0. Radiation at x=0 into a Medium at Zero.

We start with the solution for a unit source at x' in the infinite solid, namely

$$v_0=\frac{1}{2\sqrt{(\pi\kappa t)}}e^{-\frac{(x-x')^2}{4\kappa t}}.$$

Since

$$\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}},*$$

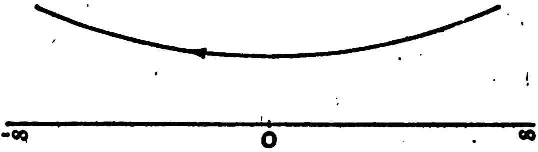
it is clear that

$$\frac{1}{2\sqrt{(\pi\kappa t)}}e^{-\frac{(x-x')^2}{4\kappa t}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\kappa\alpha^2 t}e^{\pm i\alpha(x-x')}d\alpha. \quad\text{...........(1)}$$

Consider the integral

$$\frac{1}{2\pi}\int e^{-\kappa\alpha^2 t}e^{i\alpha(x-x')}d\alpha \quad (t>0)$$

over the path (P) in the α-plane of Fig. 14, this path being



The path (P) in the α-plane

FIG. 14.

chosen so that at infinity on the right the argument of α lies between 0 and $\frac{1}{4}\pi$, and on the left between $\frac{3}{4}\pi$ and π.

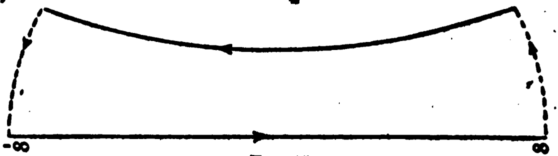


FIG. 15.

Complete this path as in Fig. 15. The integral vanishes over the circular arcs joining the path (P) to the points $\pm\infty$ on the real

* Cf. footnote, p. 30.

axis, and there are no poles of the integrand inside this closed circuit.

It follows, from Cauchy's Theorem, that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa a^2 t} e^{ia(x-x')}\,da = -\frac{1}{2\pi}\int e^{-\kappa a^2 t} e^{ia(x-x')}\,da, \quad \ldots\ldots\ldots(2)$$

the second integral being taken over the path (P).

Also we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa a^2 t} e^{-ia(x-x')}\,da = -\frac{1}{2\pi}\int e^{-\kappa a^2 t} e^{-ia(x-x')}\,da, \quad \ldots\ldots(3)$$

the second integral being taken over the path (P).

It is convenient in the following argument to take (2) when $x > x'$, and (3) when $x < x'$, for the transformation of (1).

Now $$\frac{1}{2\pi} A e^{-\kappa a^2 t} e^{iax}$$

is a solution of the differential equation $\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}$, when A is independent of x and t.

If we choose A properly, integrate this solution over the path (P), and add it to (2) and (3) above, we find a temperature function satisfying all the conditions of our problem.

The choice of A is indicated by the equation

$$-\frac{\partial v}{\partial x} + hv = 0,$$

when $x = 0$.

This equation leads to $\quad A = \dfrac{h+ia}{h-ia} e^{iax'}$.

Then we have

$$v = -\frac{1}{2\pi}\int e^{-\kappa a^2 t} e^{\pm ia(x-x')}\,da + \frac{1}{2\pi}\int \frac{h+ia}{h-ia} e^{-\kappa a^2 t} e^{ia(x+x')}\,da, \quad \ldots(4)$$

over the path (P), the positive or negative sign being chosen according as $x \gtrless x'$.

This may be written

$$v = -\frac{1}{2\pi}\int e^{-\kappa a^2 t}\left(e^{\pm ia(x-x')} + e^{ia(x+x')}\right)da + \frac{ih}{\pi}\int e^{-\kappa a^2 t} \frac{e^{ia(x+x')}}{a+ih}\,da, \quad (5)$$

the integrals being taken over the path (P) and the positive or negative sign being chosen according as $x \gtrless x'$.

We shall now show that this value of v satisfies all the conditions of our problem.

From the way in which it has been built up, we see that it satisfies the equation of conduction and the surface condition at $x=0$.

Also, from what has been said above, the first part of (5) reduces to

$$\frac{1}{2\sqrt{(\pi\kappa t)}}\left[e^{-\frac{(x-x')^2}{4\kappa t}}+e^{-\frac{(x+x')^2}{4\kappa t}}\right].$$

The first term corresponds to the source at x', the second to an equal source at $-x'$.

We have thus only to show that the second part of (5) vanishes in the limit when $t\to 0$ and x is positive.

Assuming that the integral

$$\int e^{-\kappa a^2 t}\frac{e^{ia(x+x')}}{h+ia}da, \text{ over the path } (P), \quad\ldots\ldots\ldots\ldots(6)$$

is a continuous function of t for $t=0$,* we have only to establish that

$$\int \frac{e^{ia(x+x')}}{h+ia}da, \text{ over the path } (P) \quad\ldots\ldots\ldots\ldots(7)$$

is zero.

Take the closed circuit of Fig. 16, consisting of the path (P), and the part of a circle, centre at the origin, lying above the path (P).

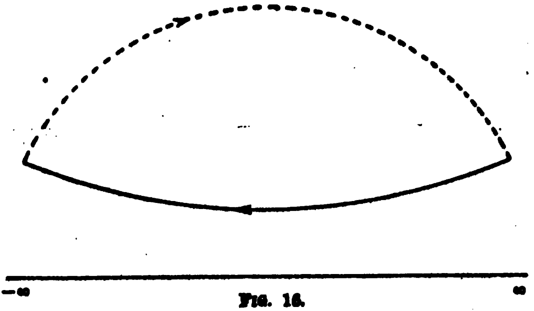


FIG. 16.

There are no poles of the integrand of (7) in this circuit, and thus the integral over the whole is zero. But when the radius tends to infinity, the integral over the circular arc vanishes, $x+x'$ being positive. Thus the integral (7) over the path (P) vanishes.

Therefore we have obtained in (5) the required temperature.

* The continuity of this integral, when $t\geqq 0$, is not difficult to establish somewhat on the lines of *F.S.*, Chapter VI. But the discussion of infinite integrals, when the variable is complex, lies somewhat outside the range of this book.

Replacing the integrals over the path (P) in (5) by the equivalent integrals over the real axis, we have

$$v=\frac{1}{\pi}\int_0^\infty e^{-\kappa a^2 t}\left\{\cos a(x-x')+\cos a(x+x')\right.$$
$$\left.-\frac{2h}{a^2+h^2}(h\cos a(x+x')-a\sin a(x+x')\right\}da. \quad\ldots\ldots(8)$$

Therefore

$$v=\frac{2}{\pi}\int_0^\infty e^{-\kappa a^2 t}\frac{(a\cos ax+h\sin ax)(a\cos ax'+h\sin ax')}{a^2+h^2}da. \quad\ldots(9)$$

Again,

$$\int_0^\infty e^{-h\xi}\cos a\xi\,d\xi=\frac{h}{a^2+h^2} \quad\text{and}\quad \int_0^\infty e^{-h\xi}\sin a\xi\,d\xi=\frac{a}{a^2+h^2}.$$

It follows that

$$\frac{2h}{\pi}\int_0^\infty e^{-\kappa a^2 t}\frac{h\cos a(x+x')-a\sin a(x+x')}{a^2+h^2}da$$
$$=\frac{2h}{\pi}\int_0^\infty e^{-\kappa a^2 t}\left[\int_0^\infty e^{-h\xi}\cos a(x+x'+\xi)\,d\xi\right]da$$
$$=\frac{2h}{\pi}\int_0^\infty e^{-h\xi}\left[\int_0^\infty e^{-\kappa a^2 t}\cos a(x+x'+\xi)\,da\right]d\xi$$
$$=\frac{h}{\sqrt{(\pi\kappa t)}}\int_0^\infty e^{-h\xi}\,e^{-\frac{(x+x'+\xi)^2}{4\kappa t}}\,d\xi.$$

Thus from (8) we have

$$v=\frac{1}{\sqrt{(\pi\kappa t)}}\left[e^{-\frac{(x-x')^2}{4\kappa t}}+e^{-\frac{(x+x')^2}{4\kappa t}}-2h\int_0^\infty e^{-h\xi}\,e^{-\frac{(x+x'+\xi)^2}{4\kappa t}}\,d\xi\right]. \quad(10)$$

The last term in (10) represents the temperature due to a line of sinks extending from $-x'$ to $-\infty$.*

88. The Same Solid. Initial Temperature f(x). Radiation into Medium at Temperature ϕ(t).

The Green's function, or the temperature at (x, y, z) at the time due to the unit source at (x', y', z') at the time τ, radiation takin

* The solution in (10) was first given by Bryan (*Cambridge, Proc. Phil. Soc.* 7, 1891). See also Bryan, *London, Proc. Math. Soc.*, 22, p. 424, 1891. The treatment in the text is taken from the author's paper, *Edinburgh, Proc. Math. Soc.* 21, 1902.

place at the boundary $x=0$ into a medium at zero, follows from § 82 (10) in the form

$$\frac{1}{2\sqrt{(\pi\kappa(t-\tau))}}\left[e^{-\frac{(x-x')^2}{4\kappa(t-\tau)}}+e^{-\frac{(x+x')^2}{4\kappa(t-\tau)}}-2h\int_0^\infty e^{-h\xi-\frac{(x+x'+\xi)^2}{4\kappa(t-\tau)}}\,d\xi\right].$$

Thus from § 80 (3), for the general problem with initial temperature $f(x)$ and the medium at temperature $\phi(t)$, the temperature at (x, y, z) at the time t is given by

$$\left.\begin{aligned} v=&\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^\infty\left[e^{-\frac{(x-x')^2}{4\kappa t}}+e^{-\frac{(x+x')^2}{4\kappa t}}-2h\int_0^\infty e^{-h\xi-\frac{(x+x'+\xi)^2}{4\kappa t}}\,d\xi\right]f(x')\,dx'\\ &+h\sqrt{\left(\frac{\kappa}{\pi}\right)}\int_0^t\left[e^{-\frac{x^2}{4\kappa(t-\tau)}}-h\int_0^\infty e^{-h\xi-\frac{(x+\xi)^2}{4\kappa(t-\tau)}}\,d\xi\right]\frac{\phi(\tau)}{\sqrt{(t-\tau)}}\,d\tau.\end{aligned}\right\}\quad(1)$$

Let the initial temperature be unity and the medium at zero. Then, from (1), we have

$$v=\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^\infty\left[e^{-\frac{(x-x')^2}{4\kappa t}}+e^{-\frac{(x+x')^2}{4\kappa t}}-2h\int_0^\infty e^{-h\xi-\frac{(x+x'+\xi)^2}{4\kappa t}}\,d\xi\right]dx'.$$

Now

$$\begin{aligned}\frac{h}{\sqrt{(\pi\kappa t)}}\int_0^\infty dx'\int_0^\infty e^{-h\xi-\frac{(x+x'+\xi)^2}{4\kappa t}}\,d\xi&=\frac{h}{\sqrt{(\pi\kappa t)}}\int_0^\infty d\xi\int_0^\infty e^{-h\xi-\frac{(x+x'+\xi)^2}{4\kappa t}}\,dx'\\ &=\frac{2h}{\sqrt{\pi}}\int_0^\infty e^{-h\xi}\left[\int_{\frac{x+\xi}{2\sqrt{(\kappa t)}}}^\infty e^{-u^2}\,du\right]d\xi\\ &=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{(\kappa t)}}}^\infty e^{-u^2}\,du-\frac{1}{\sqrt{(\pi\kappa t)}}\int_0^\infty e^{-h\xi-\frac{(x+\xi)^2}{4\kappa t}}\,d\xi.\end{aligned}$$

Thus we have

$$\begin{aligned}v&=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{(\kappa t)}}}^\infty e^{-u^2}\,du-\frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{(\kappa t)}}}^\infty e^{-u^2}\,du+\frac{1}{\sqrt{(\pi\kappa t)}}\int_0^\infty e^{-h\xi-\frac{(x+\xi)^2}{4\kappa t}}\,d\xi\\ &=\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{(\kappa t)}}}e^{-u^2}\,du+\frac{1}{\sqrt{(\pi\kappa t)}}\int_0^\infty e^{-h\xi-\frac{(x+\xi)^2}{4\kappa t}}\,d\xi,\quad\text{as in § 25.}\end{aligned}$$

84. Finite Solid. Source at x′ at t=0. Boundaries x=0 and x=a at Zero.

To obtain the Green's function when there is no radiation at the surface, we start with the solution

$$v_0=\frac{1}{2\sqrt{(\pi\kappa t)}}\left[e^{-\frac{(x-x')^2}{4\kappa t}}-e^{-\frac{(x+x')^2}{4\kappa t}}\right],$$

which satisfies the condition for a source at x' at the time $t=0$, and also the boundary condition at $x=0$.

This may be replaced as in § 82 by

$$v_0=-\frac{1}{2\pi}\int e^{-\kappa\alpha^2 t}e^{\pm i\alpha(x-x')}\,d\alpha+\frac{1}{2\pi}\int e^{-\kappa\alpha^2 t}e^{i\alpha(x+x')}\,d\alpha,$$

the integrals being taken over the standard path (P) of Fig. 14; also when $x>x'$, we choose the positive sign; when $x<x'$, the negative sign.

Thus we have the transformation

$$\left.\begin{aligned} v_0&=-\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\sin\alpha x'\,e^{i\alpha x}\,d\alpha \qquad (x>x')\\ &=-\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\sin\alpha x\,e^{i\alpha x'}\,d\alpha, \qquad (x<x')\end{aligned}\right\}\text{..............(1)}$$

the integrals being taken over the path (P).

Let $$v_1=\frac{1}{i\pi}\int A e^{-\kappa\alpha^2 t}\sin\alpha x\,d\alpha,$$

over the path (P), and choose the constant A so that the condition at $x=a$ is satisfied by $v=v_0+v_1$.

Then $$v_1=\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x\sin\alpha x'}{\sin\alpha a}e^{i\alpha a}\,d\alpha.\text{..................(2)}$$

Consider the solution

$$\left.\begin{aligned} v&=v_0+v_1\\ &=-\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x'\sin\alpha(a-x)}{\sin\alpha a}\,d\alpha \qquad (x>x')\\ &=-\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x\sin\alpha(a-x')}{\sin\alpha a}\,d\alpha. \qquad (x<x')\end{aligned}\right\}\text{...........(3)}$$

The value of v given in (3) satisfies the equation of conduction and vanishes when $x=0$ and $x=a$.

We shall now show that it satisfies the initial condition for a source at x'. We have seen that $\underset{t\to 0}{\mathrm{Lt}}\, v_0$ has the required value. We have thus only to show that $\underset{t\to 0}{\mathrm{Lt}}\, v_1=0$, where v_1 is given in (2).

Assuming that the integral

$$\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha a\sin\alpha x'}{\sin\alpha a}e^{i\alpha a}\,d\alpha, \text{ over the path } (P)\text{.............(4)}$$

is a continuous function of t for $t=0$,* we have only to establish that

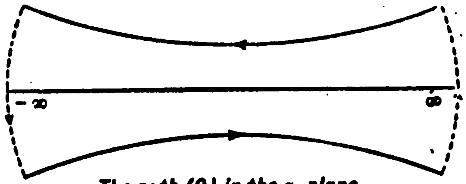
$$\int \frac{\sin \alpha x \sin \alpha x'}{\sin \alpha a} e^{i\alpha a} d\alpha, \text{ over the path } (P), \quad \text{.................}(5)$$

is zero.

This follows, as before, from the closed circuit of Fig. 16, since there are no poles of the integrand of (5) in this circuit, and the integral over the whole vanishes. But when the radius tends to infinity, the integral over the circular arc vanishes, provided that $x+x'-2a$ is negative. Thus the integral (5) over the complete path (P) is zero.

The solution (3) can be reduced to an infinite series as follows :—

Take the path (Q) of Fig. 17 formed by the path (P), the image of this path on the real axis, and the circular arcs (dotted in the



The path (Q) in the α-plane

FIG. 17.

diagram) joining the ends of these two curves. The dotted part of this circuit gives zero in the limit. Also the integrand in (3) is an odd function of α.

Thus we have from (3)

$$\left.\begin{aligned} v &= -\frac{1}{2i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin \alpha x' \sin \alpha(a-x)}{\sin \alpha a}\,d\alpha, \quad (x' \leqq x \leqq a) \\ v &= -\frac{1}{2i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin \alpha x \sin \alpha(a-x')}{\sin \alpha a}\,d\alpha. \quad (0 \leqq x \leqq x') \end{aligned}\right\} \quad \text{.......}(6)$$

the integrals now being taken over the complete path (Q).

The poles of the integrand are at $\alpha = \pm\pi/a, \pm 2\pi/a$, etc.

Thus, by Cauchy's Theorem, we obtain

$$v = \frac{2}{a}\sum_{1}^{\infty} \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}x'\, e^{-\kappa\frac{n^2\pi^2}{a^2}t}$$

for the two integrals of (6).

---

* See footnote, p. 175.

Hence the Green's function for this case is

$$\frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}x' \, e^{-\kappa\frac{n^2\pi^2}{a^2}(t-\tau)},$$

and the temperature at x at the time t, when the initial temperature is $f(x)$, and the surfaces $x=0$ and $x=a$ are kept at $\phi_1(t)$ and $\phi_2(t)$, follows from § 80 (3) in the form

$$v=\frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x\int_0^a \sin\frac{n\pi}{a}x' \, e^{-\kappa\frac{n^2\pi^2}{a^2}t} f(x')\,dx'$$

$$+\frac{2\kappa\pi}{a^2}\sum_1^\infty n\sin\frac{n\pi}{a}x\int_0^t [\phi_1(\tau)-(-1)^n\phi_2(\tau)]\,e^{-\kappa\frac{n^2\pi^2}{a^2}(t-\tau)}d\tau.*$$

85. The Same Solid. Source at x′ at t=0. Radiation at x=0 and x=a into a Medium at Zero.

Here we start with the solution

$$v_0=\frac{1}{2\sqrt{(\pi\kappa t)}}e^{-\frac{(x-x')^2}{4\kappa t}}$$

$$=-\frac{1}{2\pi}\int e^{-\kappa\alpha^2 t}e^{i\alpha(x-x')}d\alpha, \quad (x>x')$$

$$=-\frac{1}{2\pi}\int e^{-\kappa\alpha^2 t}e^{-i\alpha(x-x')}d\alpha, \quad (x<x')$$

the integrals being taken over the standard path (P) of Fig. 14.

We associate with this another solution,

$$v_1=\frac{1}{2\pi}\int e^{-\kappa\alpha^2 t}(Ae^{i\alpha x}+Be^{-i\alpha x})\,d\alpha,$$

over the path (P), and determine A and B so that the boundary conditions

$$\mp\frac{\partial v}{\partial x}+hv=0, \quad \text{when } x=0 \text{ and } x=a,$$

are satisfied by $v=v_0+v_1$.

In this way we obtain

$$A=(h+i\alpha)\frac{(h\sin\alpha(a-x')+\alpha\cos\alpha(a-x')}{(h^2-\alpha^2)\sin\alpha a+2\alpha h\cos\alpha a},$$

$$B=(h+i\alpha)\frac{h\sin\alpha x'+\alpha\cos\alpha x'}{(h^2-\alpha^2)\sin\alpha a+2\alpha h\cos\alpha a}e^{i\alpha a}.$$

---

* Cf. § 34.

Substituting these values for A and B in v_1, we have finally

$$v_0 = v_e + v_1$$

$$= \frac{i}{\pi}\int e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x' + \alpha\cos\alpha x')(h\sin\alpha(a-x) + \alpha\cos\alpha(a-x))}{(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a}\,d\alpha, \quad (1)$$

when $x > x'$; and, when $x < x'$, we interchange x and x' in this expression, the integral being taken over the path (P).

From the way in which (1) has been built up, we know that it satisfies the equation of conduction and the boundary conditions. We shall now show that it satisfies the initial conditions for a source at x'.

As v_e corresponds to the source at x', we have only to show that $\operatorname*{Lt}_{t\to 0} v_1 = 0$.

This follows as in § 84, by introducing the path of Fig. 16, for the integrals vanish over the circular arc in the limit, provided that

$$x + x' > 0,$$

and

$$x + x' - 2a < 0,$$

and both of these conditions are satisfied.

Also we know that the roots of the equation

$$(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a = 0$$

are all real and not repeated. (Cf. § 36.)

The solution in (1) can now be reduced to an infinite series by using the path (Q) of Fig. 17. For we have from (1), when $x > x'$,

$$v = -\frac{1}{2i\pi}\int e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x' + \alpha\cos\alpha x')(h\sin\alpha(a-x) + \alpha\cos\alpha(a-x))}{(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a}\,d\alpha, \quad (2)$$

the integral being taken over the path (Q).

Then, by Cauchy's Theorem, we have

$$v = 2\sum_{\alpha} e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x' + \alpha\cos\alpha x')(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}, \quad \text{.........}(3)$$

the summation being taken over the positive roots of the equation

$$(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a = 0.$$

The result in (3) holds both when $x > x'$ and when $x < x'$.

Hence the Green's function is

$$2\sum_{\alpha} e^{-\kappa\alpha^2(t-\tau)}\frac{(h\sin\alpha x' + \alpha\cos\alpha x')(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}.$$

The solution for an arbitrary initial temperature $f(x)$ follows at once, and we obtain the temperature at x at the time t, for the case when the medium is at zero, in the form

$$v=2\int_0^a f(x')\sum_\alpha e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x'+\alpha\cos\alpha x')(h\sin\alpha x+\alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}\,dx'. \quad (4)$$

This admits of term by term integration, and may be written

$$v=2\sum_\alpha e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x+\alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}\int_0^a f(x')(h\sin\alpha x'+\alpha\cos\alpha x')\,dx'. \quad (5)$$

(Cf. § 36.)

It may be noted that the result given in (5) leads to the expansion:

$$f(x)=2\,\underset{t\to 0}{\mathrm{Lt}}\sum_\alpha e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x+\alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}\int_0^a f(x')(h\sin\alpha x'+\alpha\cos\alpha x')\,dx'.$$

Ex. 1. The same Solid. Source at x'. The boundary $x=0$ kept at zero temperature. The boundary $x=a$ radiating into medium at zero.

Result: $$v=2\sum_\alpha e^{-\kappa\alpha^2 t}\sin\alpha x\sin\alpha x'\frac{\alpha^2+h^2}{a\alpha^2+h(1+ha)},$$

where the summation is taken over the positive roots of the equation

$$\alpha\cos\alpha a+h\sin\alpha a=0.$$

Ex. 2. The same Solid. Boundary conditions as above. Initial temperature $f(x)$.

Result: $$v=2\sum_\alpha e^{-\kappa\alpha^2 t}\frac{\alpha^2+h^2}{a\alpha^2+h(1+ha)}\sin\alpha x\int_0^a \sin\alpha x' f(x')\,dx'.$$

Ex. 3. The same Solid. Source at x'. The boundary $x=0$ impervious to heat. The boundary $x=a$ radiating into a medium at zero temperature.

Result: $$v=2\sum_\alpha e^{-\kappa\alpha^2 t}\frac{\alpha^2+h^2}{a\alpha^2+h(1+ha)}\cos\alpha x\cos\alpha x',$$

where the summation extends over the positive roots of

$$\alpha\sin\alpha a-h\cos\alpha a=0.$$

Ex. 4. The same Solid. Boundary conditions as above. Initial temperature $f(x)$.

Result: $$v=2\sum_\alpha e^{-\kappa\alpha^2 t}\cos\alpha x\frac{\alpha^2+h^2}{a\alpha^2+h(1+ha)}\int_0^a \cos\alpha x' f(x')\,dx'.$$

86. Two-Dimensional Problems.*

I. *Semi-Infinite Solid* $y>0$. *Initial Temperature* $f(x, y)$. *Boundary* $y=0$ *kept at* $F(x, t)$.

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* Cf. Hobson, *London, Proc. Math. Soc.*, 19, pp. 286, 293, 1889, for the result given in §§ 86, 87.

In this case the Green's function is

$$u=\frac{1}{4\pi\kappa(t-\tau)}\left(e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa(t-\tau)}}-e^{-\frac{(x-x')^2+(y+y')^2}{4\kappa(t-\tau)}}\right)$$

and $\left(\frac{\partial u}{\partial n}\right)_t=\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{y'}{4\pi\kappa^2(t-\tau)^2}e^{-\frac{(x-x')^2+y'^2}{4\kappa(t-\tau)}}.$

It follows from § 80 (4) that the temperature at (x, y) at the time t is given by

$$v=\frac{1}{4\kappa\pi t}\int_{-\infty}^{\infty}\int_0^{\infty}f(x', y')\left[e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa t}}-e^{-\frac{(x-x')^2+(y+y')^2}{4\kappa t}}\right]dx'\,dy'$$

$$+\frac{y}{4\pi\kappa}\int_0^t\int_{-\infty}^{\infty}\frac{F(x', \tau)}{(t-\tau)^2}e^{-\frac{(x-x')^2+y^2}{4\kappa(t-\tau)}}d\tau\,dx'.*$$

II. *The Same Solid. Radiation at $y=0$ into Medium at $F(x, t)$.*

In this case the Green's function may be deduced from § 82 in the form

$$u=\frac{1}{4\pi\kappa(t-\tau)}\left[e^{-\frac{(x-x')^2+(y-y')^2}{4\kappa(t-\tau)}}+e^{-\frac{(x-x')^2+(y+y')^2}{4\kappa(t-\tau)}}\right.$$
$$\left.-2h\int_0^{\infty}e^{-h\eta}e^{-\frac{(x-x')^2+(y+y'+\eta)^2}{4\kappa(t-\tau)}}d\eta\right]$$

and

$$\left(\frac{\partial u}{\partial n}\right)_t=\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{h}{4\pi\kappa^2(t-\tau)^2}\int_0^{\infty}e^{-h\eta}e^{-\frac{(x-x')^2+(y'+\eta)^2}{4\kappa(t-\tau)}}(y'+\eta)\,d\eta.$$

Thus the temperature at (x, y) at the time t, when the initial temperature is zero, is given by

$$v=\frac{h}{4\pi\kappa}\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F(x', \tau)}{(t-\tau)^2}(y+\eta)e^{-h\eta-\frac{(x-x')^2+(y+\eta)^2}{4\kappa(t-\tau)}}d\tau\,dx'\,d\eta.$$

87. Three-Dimensional Problems.

I. *Semi-Infinite Solid $x>0$. Initial Temperature $f(x, y, z)$. Boundary $x=0$ kept at $F(y, z, t)$.*

In this case the Green's function is

$$u=\frac{1}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}}\left(e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}-e^{-\frac{(x+x')^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}\right)$$

* This result proves the truth of the statement at the end of § 74 about continuous doublets.

and $$\left(\frac{\partial u}{\partial n}\right)_t=\left(\frac{\partial u}{\partial x}\right)_{x=0}=\frac{x'}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}(t-\tau)^{\frac{5}{2}}}e^{-\frac{x'^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}.$$

Thus the temperature at (x, y, z) at the time t is given by

$$v=\frac{1}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}t^{\frac{3}{2}}}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(x', y', z')$$
$$\times\left[e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}-e^{-\frac{(x+x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}\right]dx'\,dy'\,dz'$$
$$+\frac{x}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}}\int_0^t\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{F(y', z', \tau)}{(t-\tau)^{\frac{5}{2}}}e^{-\frac{x^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}d\tau\,dy'\,dz'.^*$$

II. *Same Solid. Radiation at $x=0$ into Medium at $F(y, z, t)$.*

The Green's function is in this case

$$u=\frac{1}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}}\Bigg[e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}+e^{-\frac{(x+x')^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}$$
$$-2h\int_0^\infty e^{-h\xi}e^{-\frac{(x+x'+\xi)^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}d\xi\Bigg].$$

Thus the temperature at (x, y, z) at the time t, when the initial temperature is zero, is given by

$$v=\frac{h}{8\pi^{\frac{3}{2}}\kappa^{\frac{3}{2}}}\int_0^t\int_{-\infty}^\infty\int_{-\infty}^\infty\int_0^\infty\frac{F(y', z', \tau)}{(t-\tau)^{\frac{5}{2}}}(x+\xi)e^{-h\xi-\frac{(x+\xi)^2+(y-y')^2+(z-z')^2}{4\kappa(t-\tau)}}$$
$$\times d\tau\,dy'\,dz'\,d\xi.$$

83. Infinite Cylinder r=a. Initial Temperature f(r, θ). Surface Temperature Zero.

To obtain the Green's function for this case, we start with

$$v_0=\frac{1}{4\pi\kappa t}e^{-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{4\kappa t}}.$$

We transform this, as in § 78 (III.), into

$$v_0=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int_0^\infty \alpha e^{-\kappa\alpha^2 t}J_n(\alpha r')J_n(\alpha r)\,d\alpha.$$

* The footnote on page 183 also applies to this result.

But

$$\int_0^\infty \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')J_n(\alpha r)\,d\alpha$$

$$=\tfrac{1}{2}\int_0^\infty \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')\{H_n^{(1)}(\alpha r)-e^{in\pi}H_n^{(1)}(-\alpha r)\}\,d\alpha^*$$

$$=\tfrac{1}{2}\int_{-\infty}^\infty \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')H_n^{(1)}(\alpha r)\,d\alpha.$$

And proceeding as in § 82, this may be replaced by

$$\left.\begin{aligned}\int_0^\infty \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')J_n(\alpha r)\,d\alpha &= -\tfrac{1}{2}\int \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')H_n^{(1)}(\alpha r)\,d\alpha \quad (r>r')\\ &= -\tfrac{1}{2}\int \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r)H_n^{(1)}(\alpha r')\,d\alpha, \quad (r<r')\end{aligned}\right\}$$

the integrals being taken over the standard path (P) of Fig. 14.

Thus we have

$$v_0=-\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')H_n^{(1)}(\alpha r)\,d\alpha, \quad \ldots\ldots(1)$$

when $r>r'$, and we interchange r and r', when $r<r'$, the integrals being taken over the path (P).

To satisfy the condition at $r=a$, we associate with v_0 another solution v_1 taken over the same path (P), where

$$v_1=\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n\,(\theta-\theta')\int A\alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')J_n(\alpha r)\,d\alpha,$$

and we choose the constant A so that v_0+v_1 shall be zero when $r=a$.

Then we have

$$v_1=\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int \alpha e^{-\kappa\alpha^2 t} J_n(\alpha r')J_n(\alpha r)\,\frac{H_n^{(1)}(\alpha a)}{J_n(\alpha a)}\,d\alpha. \quad \ldots(2)$$

Consider the solution

$$\begin{aligned}v&=v_0+v_1\\ &=-\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int \alpha e^{-\kappa\alpha^2 t}\frac{J_n(\alpha r')}{J_n(\alpha a)}\\ &\qquad\times\{H_n^{(1)}(\alpha r)J_n(\alpha a)-H_n^{(1)}(\alpha a)J_n(\alpha r)\}\,d\alpha, \quad \ldots\ldots(3)\end{aligned}$$

---

* When x is real and positive, we know from Appendix I., § 4 that

$$H_n^{(1)}(x)=\frac{i}{\sin n\pi}\left(e^{-ni\pi}J_n(x)-J_{-n}(x)\right)$$

and

$$H_n^{(1)}(-x)=H_n^{(1)}(xe^{i\pi})=\frac{i}{\sin n\pi}\left(J_n(x)-e^{-ni\pi}J_{-n}(x)\right).$$

Thus

$$H_n^{(1)}(x)-e^{ni\pi}H_n^{(1)}(-x)=2J_n(x).$$

when $r > r'$, the integral being taken over the path (P); and r, r' being interchanged, when $r < r'$.

The value of v given in (3) satisfies the equation of conduction, and vanishes when $r=a$.

We shall now show that it satisfies the initial condition for a line source at (r', θ'). This requires that $\operatorname*{Lt}_{t\to 0} v_1 = 0$, where v_1 is given in (2).

But this follows as in § 84 by introducing the path of Fig. 16. From the approximations for $J_n(z)$ and $H_n^{(1)}(z)$ in the upper part of the z-plane,* it will be seen that

$$\int \alpha \frac{J_n(\alpha r')J_n(\alpha r)H_n^{(1)}(\alpha a)}{J_n(\alpha a)}\, d\alpha$$

vanishes over the circular arc in the limit, provided that $r + r' - 2a < 0$, a condition which is satisfied.

Also we know that the roots of $J_n(\alpha a)=0$ are all real and not repeated.

The solution in (3) can now be reduced to an infinite series by using the path (Q) of Fig. 17.

For we may replace the term

$$-\frac{1}{4\pi}\int \alpha e^{-\kappa\alpha^2 t}\frac{J_n(\alpha r')}{J_n(\alpha a)}\{H_n^{(1)}(\alpha r)J_n(\alpha a) - H_n^{(1)}(\alpha a)J_n(\alpha r)\}\, d\alpha$$

over the path (P), by half this integral over the path (Q).

Using Cauchy's Theorem, this term becomes

$$\frac{i}{2a}\sum_{\alpha} \alpha e^{-\kappa\alpha^2 t}\frac{J_n(\alpha r')J_n(\alpha r)H_n^{(1)}(\alpha a)}{J_n'(\alpha a)},$$

the summation being taken over the positive roots of $J_n(\alpha a)=0$.

But it is known that †

$$J_n(z)\frac{d}{dz}H_n^{(1)}(z) - H_n^{(1)}(z)\frac{d}{dz}J_n(z) = \frac{2i}{\pi z}.$$

Thus we find for the temperature at (r, θ) at the time t, due to the source at (r', θ'),

$$v = \frac{1}{\pi a^2}\sum_{-\infty}^{\infty}\sum_{\alpha}\cos n(\theta-\theta')\, e^{-\kappa\alpha^2 t}\frac{J_n(\alpha r')J_n(\alpha r)}{[J_n'(\alpha a)]^2}, \quad \text{.........(4)}$$

and this holds when $r \gtrless r'$.

---

* Cf. Appendix I., §§ 4, 5.

† Cf. Watson, *loc. cit.*, § 3. 63. (1).

The Green's function for this case is, therefore,

$$\frac{1}{\pi a^2}\sum_{-\infty}^{\infty}\sum_{a}\cos n(\theta-\theta')\,e^{-\kappa a^2(t-\tau)}\frac{J_n(ar')J_n(ar)}{[J_n'(aa)]^2}.$$

When the initial temperature is $f(r)$, we obtain the temperature at (r, θ) at the time t, in the form

$$v=\frac{2}{a^2}\int_0^a r'f(r')\sum_a e^{-\kappa a^2 t}\frac{J_0(ar)J_0(ar')}{[J_0'(aa)]^2}\,dr'$$

$$=\frac{2}{a^2}\sum_a e^{-\kappa a^2 t}\frac{J_0(ar)}{[J_0'(aa)]^2}\int_0^a r'f(r')J_0(ar')\,dr'.$$

When the initial temperature is $f(r, \theta)$, we have

$$v=\frac{1}{\pi a^2}\int_0^a\int_{-\pi}^{\pi} r'f(r',\theta')\left[\sum_{-\infty}^{\infty}\sum_a \cos n(\theta-\theta')e^{-\kappa a^2 t}\frac{J_n(ar)J_n(ar')}{[J_n'(aa)]^2}\right]dr'\,d\theta'.$$

If we assume that this series may be integrated term by term, we have, for the coefficient of $J_n(ar)\cos n\theta$ the expression

$$\frac{2}{\pi a^2}\frac{e^{-\kappa a^2 t}}{[J_n'(aa)]^2}\int_0^a\int_{-\pi}^{\pi} r'f(r',\theta')J_n(ar')\cos n\theta'\,dr'\,d\theta'$$

and for $n=0$ the result must be divided by two.

Thus we are led to the series for $f(r)$ and $f(r, \theta)$ of § 57, I. and IV., and they occur here as the limiting values of v when $t\to 0$.

89. Infinite Cylinder $r=a$. Radiation at the Surface into a Medium at Zero. Initial Temperature $f(r, \theta)$. $v_i = v_f = \text{const}$

Starting with the expression for the line source at (r', θ'), we transform it as before into

$$\left.\begin{aligned} v_0&=-\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int a e^{-\kappa a^2 t}J_n(ar')H_n^{(1)}(ar)\,da \quad (r>r')\\ &=-\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int a e^{-\kappa a^2 t}J_n(ar)H_n^{(1)}(ar')\,da, \quad (r<r'),\end{aligned}\right\}\quad\ldots(1)$$

the integrals being taken over the standard path (P) of Fig. 14.

We then obtain the solution

$$v_1=\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int a e^{-\kappa a^2 t}J_n(ar)J_n(ar')$$

$$\times\frac{\left(a\dfrac{d}{d(aa)}H_n^{(1)}(aa)+hH_n^{(1)}(aa)\right)}{(aJ_n'(aa)+hJ_n(aa))}\,da\quad\ldots(2)$$

over the path (P), and we prove that $v=v_0+v_1$, which satisfies the

surface condition $\dfrac{\partial v}{\partial r}+hv=0$, also satisfies the initial condition for a source at (r', θ'). we have 2nd order

Thus we are led to the solution

$$\begin{aligned} v &= v_0+v_1 \\ &= -\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int \alpha e^{-\kappa\alpha^2 t}J_n(\alpha r') \\ &\quad \times \frac{\left\{H_n^{(1)}(\alpha r)[\alpha J_n'(\alpha a)+hJ_n(\alpha a)] - J_n(\alpha r)\left[\alpha\dfrac{d}{d(\alpha a)}H_n^{(1)}(\alpha a)+hH_n^{(1)}(\alpha a)\right]\right\}}{\alpha J_n'(\alpha a)+hJ_n(\alpha a)}\,d\alpha, \quad \ldots(3) \end{aligned}$$

when $r>r'$, the integral being taken over the path (P); and when $r<r'$ we interchange r and r'.

The solution in (3) can be reduced to an infinite series by using the path (Q) of Fig. 17, for we know* that the roots of the equation

$$\alpha J_n'(\alpha a)+hJ_n(\alpha a)=0, \quad \ldots\ldots(4)$$

are all real and not repeated.

The coefficient of $\cos n(\theta-\theta')$ becomes

$$\frac{i}{2}\sum_{\alpha}\alpha e^{-\kappa\alpha^2 t}J_n(\alpha r)J_n(\alpha r')\,\frac{\alpha\dfrac{d}{d(\alpha a)}H_n^{(1)}(\alpha a)+hH_n^{(1)}(\alpha a)}{\alpha aJ_n''(\alpha a)+(1+ha)J_n'(\alpha a)},$$

the summation being taken over the positive roots of (4).

But

$$J_n(\alpha a)\frac{d}{d(\alpha a)}H_n^{(1)}(\alpha a)-H_n^{(1)}(\alpha a)\frac{d}{d(\alpha a)}J_n(\alpha a)=\frac{2i}{\pi\alpha a}$$

and

$$\alpha\frac{d}{d(\alpha a)}J_n(\alpha a)+hJ_n(\alpha a)=0.$$

Therefore
$$\alpha\frac{d}{d(\alpha a)}H_n^{(1)}(\alpha a)+hH_n^{(1)}(\alpha a)=\frac{2i}{\pi aJ_n(\alpha a)}.$$

Also we find that

$$\alpha aJ_n''(\alpha a)+(1+ha)J_n'(\alpha a)=-\frac{a}{\alpha}\left(h^2+\alpha^2-\frac{n^2}{a^2}\right)J_n(\alpha a).$$

Therefore the coefficient of $\cos n(\theta-\theta')$ becomes

$$\frac{1}{\pi a^2}\sum_{\alpha}\alpha^2 e^{-\kappa\alpha^2 t}\,\frac{J_n(\alpha r)J_n(\alpha r')}{\left(h^2+\alpha^2-\dfrac{n^2}{a^2}\right)(J_n(\alpha a))^2},$$

the summation being taken over the positive roots of (4).

---

* Cf. footnote, p. 117.

Hence the temperature at (r, θ) at the time t due to a source at (r', θ'), when radiation takes place at the surface into a medium at zero, is given by

$$v=\frac{1}{\pi a^2}\sum_{-\infty}^{\infty}\sum_{\alpha}\cos n(\theta-\theta')\alpha^2 e^{-\kappa\alpha^2 t}\frac{J_n(\alpha r)J_n(\alpha r')}{\left(h^2+\alpha^2-\frac{n^2}{a^2}\right)(J_n(\alpha a))^2}, \quad \ldots(5)$$

and this holds when $r \gtrless r'$.

When the initial temperature is $f(r)$, or $f(r, \theta)$, we have

$$v=\frac{2}{a^2}\sum_{\alpha}\alpha^2 e^{-\kappa\alpha^2 t}\frac{J_0(\alpha r)}{(h^2+\alpha^2)(J_0(\alpha a))^2}\int_0^a r' f(r')J_0(\alpha r')\,dr' \quad \ldots\ldots(6)$$

and

$$v=\frac{1}{\pi a^2}\int_0^a\int_{-\pi}^{\pi} r' f(r', \theta')$$

$$\times\left[\sum_{-\infty}^{\infty}\sum_{\alpha}\cos n(\theta-\theta')\alpha^2 e^{-\kappa\alpha^2 t}\frac{J_n(\alpha r)J_n(\alpha r')}{\left(h^2+\alpha^2-\frac{n^2}{a^2}\right)(J_n(\alpha a))^2}\right]dr'\,d\theta'. \quad \ldots(7)$$

If we assume that the series (7) can be integrated term by term, the coefficient of $J_n(\alpha r)\cos n\theta$ is

$$\frac{2\alpha^2 e^{-\kappa\alpha^2 t}}{\pi a^2\left(h^2+\alpha^2-\frac{n^2}{a^2}\right)(J_n(\alpha a))^2}\int_0^a\int_{-\pi}^{\pi} r' f(r', \theta')J_n(\alpha r')\cos n\theta'\,dr'\,d\theta',$$

and for $n=0$, the result must be halved.

Thus we are led to the expansion for $f(r)$ and $f(r, \theta)$ of § 57, III. and V., and they occur here as the limiting values of v when $t \to 0$.

90. The Wedge of any Angle.*

In this section we shall find the temperature due to a unit line source at (r', θ') at $t=0$ in the wedge formed by $\theta=0$, $\theta=\theta_0$, these planes being kept at zero. When the wedge has an initial temperature $f(r, \theta)$, and the planes are kept at zero, the temperature at the time t follows at once from the solution obtained below.

I. We start with the solution

$$v_0=\frac{1}{4\pi\kappa t}e^{-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{4\kappa t}}.$$

* Cf. Carslaw, *London, Proc. Math. Soc.* (Ser. 2), 8, p. 365, 1910.

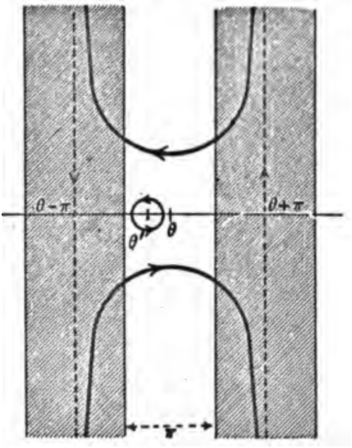
Introducing the complex variable α, this becomes

$$v_\theta = \frac{1}{8\pi^2\kappa t} e^{-\frac{r^2+r'^2}{4\kappa t}} \int e^{\frac{rr'}{2\kappa t}\cos(\alpha-\theta)} \frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\, d\alpha, \quad \ldots\ldots\ldots\ldots$$

the integral being taken over any closed path in the α-pla
enclosing the point $\alpha=\theta'$ and no other singularity of the integran

It is clear that singularities enter only from the poles $\alpha=2m\pi+$
and the infinities of $e^{[rr'\cos(\alpha-\theta)]/2\kappa t}$.

On putting $\alpha=a+ib$, we see that when $b=\pm\infty$, $\cos(\alpha-$
must be negative or the integrand would be infinite. Hence i
deforming the path to $b=\pm\infty$, we must take care to have α in suc
a region that $\cos(\alpha-\theta)$ must be negative. The shaded portions



The path (A) in the α-plane

FIG. 18.

Fig. 18 represent such parts of the α-plane, and taking $|\theta-\theta'|<\pi$
the circuit round $\alpha=\theta'$ may be deformed into that given in th
figure, this new path being composed of two symmetrical curve
parts extending to infinity, and two rectilinear parts, drawn at
distance 2π from each other. The integrals over these straigh
lines cut each other out, owing to the periodicity of the integran
and the fact that they are described in opposite directions. We ar
left with the two curved portions, which we refer to as the path (A
in the α-plane for this value of θ.

In connection with the problem of the wedge of angle $n\pi/m$, this solutio
was used to obtain another of period $2n\pi$, with only one singularity in the

range, and to the second solution the method of images was applied. (Cf. §79, and the author's paper in *Proc. London Math. Soc.*, **30**, 1899.)

Consider the expression

$$v=\frac{1}{8\pi\kappa\theta_0 t}e^{-\frac{r^2+r'^2}{4\kappa t}}\int e^{\frac{rr'}{2\kappa t}\cos(\alpha-\theta)}\frac{e^{\frac{i\pi\alpha}{\theta_0}}\,d\alpha}{e^{\frac{i\pi\alpha}{\theta_0}}-e^{\frac{i\pi\theta'}{\theta_0}}}, \qquad \text{.........(2)}$$

the integral being taken over the path (A), corresponding to the current coordinate θ.

(i) *This expression is a solution of the equation of conduction*, since every element of the integrand satisfies this equation.

(ii) *It is periodic in θ and of period $2\theta_0$.* A change in θ, *e.g.* from θ to ϕ, simply translates the path along the real axis of α, and the term $e^{rr'\cos(\alpha-\theta)/2\kappa t}$ is unaltered. Further, if the change in θ is equal to $2n\theta_0$ (n being any positive or negative integer), the other factor of the integrand also remains unaltered.

(iii) *It vanishes when $t\to 0$ in the interval $-\theta_0<\theta<\theta_0$, except when $r\to r'$ and $\theta\to\theta'$, where it takes the form*

$$\underset{t\to 0,\ r\to r',\ \theta\to\theta'}{\mathrm{Lt}}\left(\frac{e^{-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{4\kappa t}}}{4\pi\kappa t}\right).$$

To prove this, we have only to note that the path (A) can be changed into the two straight lines of Fig. 18, together with the small circuits surrounding such of the poles as lie in the interval $(\theta-\pi,\ \theta+\pi)$. The integrals over these straight lines vanish in the limit as $t\to 0$, since these lie in the shaded portions of the diagram. We are thus left with the circuits round the poles; and if $-\theta_0<\theta'<\theta_0$ and $-\theta_0<\theta<\theta_0$, the integrals round these circuits all vanish when $t\to 0$, except one which corresponds to the source at $(r',\ \theta')$.

(iv) *When $r\to\infty$, the expression vanishes* owing to the presence of the factor $e^{-r^2/4\kappa t}$.

II. The solution given in (2) can be transformed into a series involving Bessel's functions by the aid of the contour integral which defines the Bessel's function of the first kind.*

* Cf. Whittaker and Watson, *loc. cit.* (3rd Ed.), p. 363.

We start with

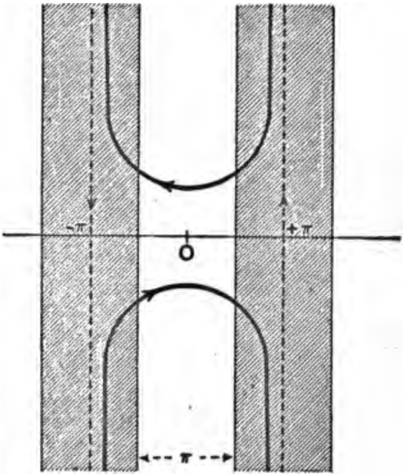
$$v=\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{8\pi\kappa\theta_0 t}\int e^{\frac{rr'}{2\kappa t}\cos(\alpha-\theta)}\frac{e^{\frac{i\pi\alpha}{\theta_0}}}{e^{\frac{i\pi\alpha}{\theta_0}}-e^{\frac{i\pi\theta'}{\theta_0}}}d\alpha$$

over the path (A).

Putting $\alpha-\theta=\alpha'$, this becomes

$$v=\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{8\pi\kappa\theta_0 t}\int e^{\frac{rr'}{2\kappa t}\cos\alpha'}\frac{e^{\frac{i\pi(\theta+\alpha')}{\theta_0}}d\alpha'}{e^{\frac{i\pi(\theta+\alpha')}{\theta_0}}-e^{\frac{i\pi\theta'}{\theta_0}}}$$

over the path (A′) of Fig. 19.



The path (A′) in the α′-plane

FIG. 19.

In expanding the term $\dfrac{e^{\frac{i\pi(\theta+\alpha')}{\theta_0}}}{e^{\frac{i\pi(\theta+\alpha')}{\theta_0}}-e^{\frac{i\pi\theta'}{\theta_0}}}$

we must proceed differently for the upper and lower parts of the path (A′).

From the upper part we obtain

$$-\sum_1^\infty e^{in\pi(\theta-\theta')/\theta_0}e^{in\pi\alpha'/\theta_0},$$

since at the positive imaginary infinity in the α'-plane $e^{in\pi\alpha'/\theta_0}$ vanishes and the series is convergent.

Also from the lower part we have

$$\sum_0^\infty e^{-in\pi(\theta-\theta')/\theta_0}e^{-in\pi\alpha'/\theta_0}.$$

Now if we change the sign of α' in the lower part, we bring this integral to the upper part, and have, finally,

$$v=-\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{8\pi\kappa\theta_0 t}\Big[\int e^{\frac{rr'}{2\kappa t}\cos\alpha'}d\alpha'+2\sum_1^\infty\cos\frac{n\pi(\theta-\theta')}{\theta_0}\int e^{\frac{rr'}{2\kappa t}\cos\alpha'}e^{\frac{in\pi\alpha'}{\theta_0}}d\alpha'\Big]$$

over the upper portion of the path (A').

Now put $u'=rr'e^{-i\alpha'}/4\kappa t$. In the u-plane, the path proceeds from the negative end of the real axis on the lower side, makes a circuit round the origin, and ends at the negative end of the real axis on the upper side, as in Fig. 20.

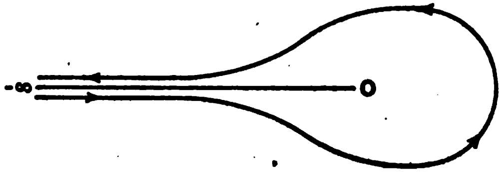


FIG. 20.

Also the integral
$$\int e^{\frac{rr'}{2\kappa t}\cos\alpha'}e^{\frac{in\pi\alpha'}{\theta_0}}d\alpha'$$

is replaced by
$$i\left(\frac{rr'}{4\kappa t}\right)^{\frac{n\pi}{\theta_0}}\int e^{u-\frac{1}{4u}(irr'/2\kappa t)^2}u^{-\left(\frac{n\pi}{\theta_0}+1\right)}du$$

over this path in the u-plane.

Thus it is replaced by

$$-2\pi e^{-\frac{in\pi^2}{2\theta_0}}J_{\frac{n\pi}{\theta_0}}\left(\frac{irr'}{2\kappa t}\right),$$

on using the contour integral for $J_n(z)$.

Hence we have

$$v=\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{4\kappa\theta_0 t}\sum_{-\infty}^{\infty}e^{-\frac{in\pi^2}{2\theta_0}}J_{\frac{n\pi}{\theta_0}}\left(\frac{irr'}{2\kappa t}\right)\cos\frac{n\pi}{\theta_0}(\theta-\theta').$$

But $\displaystyle\int_0^\infty \alpha e^{-\kappa\alpha^2 t}J_n(\alpha r)J_n(\alpha r')\,d\alpha=\frac{1}{2\kappa t}e^{-\frac{r^2+r'^2}{4\kappa t}}e^{-\frac{1}{2}in\pi}J_n\left(\frac{irr'}{2\kappa t}\right)$*

for any value of n for which the real part is greater than -1.

* Cf. Gray and Mathews, *loc. cit.*, p. 78 (161): Nielsen, *loc. cit.*, p. 184 (2). This formula for a positive integer n was first given by Hankel (*Math. Ann., Leipzig*, 8, p. 470, 1875) and for the general case by Sonine (*ibid.*, 16, p. 40, 1880).

Therefore our solution, with period $2\theta_0$, can be written

$$v=\frac{1}{2\theta_0}\sum_{-\infty}^{\infty}\cos\frac{n\pi}{\theta_0}(\theta-\theta')\int_0^{\infty}\alpha e^{-\kappa\alpha^2 t}J_{\frac{n\pi}{\theta_0}}(\alpha r)J_{\frac{n\pi}{\theta_0}}(\alpha r')\,d\alpha. \quad\text{......(3)}$$

III. Now let θ' lie within $0<\theta<\theta_0$, and denote the solution found above in (3) by $v(\theta')$.

This has the value required by a source at the points $(r', \theta'\pm 2s\theta_0)$, s being zero or any positive integer.

Similarly denote by $v(-\theta')$, the corresponding solution for $(-\theta')$.

Then $\quad v=v(\theta')-v(-\theta')$

$$=\frac{1}{2\theta_0}\sum_{-\infty}^{\infty}\left(\cos\frac{n\pi}{\theta_0}(\theta-\theta')-\cos\frac{n\pi}{\theta_0}(\theta+\theta')\right)$$

$$\times\int_0^{\infty}\alpha e^{-\kappa\alpha^2 t}J_{\frac{n\pi}{\theta_0}}(\alpha r)J_{\frac{n\pi}{\theta_0}}(\alpha r')\,d\alpha \quad\text{.........(4)}$$

satisfies all the conditions for the temperature in the wedge $0<\theta<\theta_0$, the unit line source being placed at (r', θ') at $t=0$, and the planes being kept at zero.

The solution given in (4) can be written in the form

$$v=\frac{2}{\theta_0}\sum_{1}^{\infty}\sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'\int_0^{\infty}\alpha e^{-\kappa\alpha^2 t}J_{\frac{n\pi}{\theta_0}}(\alpha r)J_{\frac{n\pi}{\theta_0}}(\alpha r')\,d\alpha. \quad\text{.....(5)}$$

In § 78, III., we found this solution by images for the wedge of angle π/m, where m is any positive integer. The result is now seen to be true for a wedge of any angle.

We have been dealing in this section with a line source through (r', θ'). If we start with a point source at (r', θ', z') and take

$$v_0=\frac{e^{-\frac{(z-z')^2}{4\kappa t}}}{(2\sqrt{(\pi\kappa t)})^3}e^{-\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{4\kappa t}},$$

we obtain by a similar argument the solution for the point source in the form:

$$v=\frac{e^{-\frac{(z-z')^2}{4\kappa t}}}{\theta_0\sqrt{(\pi\kappa t)}}\sum_{1}^{\infty}\sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'\int_0^{\infty}\alpha e^{-\kappa\alpha^2 t}J_{\frac{n\pi}{\theta_0}}(\alpha r)J_{\frac{n\pi}{\theta_0}}(\alpha r')\,d\alpha.$$

91. Infinite Cylinder. The Surface $r=a$ and the Planes $\theta=0$, $\theta=\theta_0$ kept at Zero. Initial Temperature $f(r, \theta)$.

We start with the solution of § 90, corresponding to the line source at (r', θ'), namely,

$$v_0=\frac{2}{\theta_0}\sum_{1}^{\infty}\sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'\int_0^{\infty}\alpha e^{-\kappa\alpha^2 t}J_{\frac{n\pi}{\theta_0}}(\alpha r')J_{\frac{n\pi}{\theta_0}}(\alpha r)\,d\alpha.$$

As we have seen in § 88, this may be written

$$\left.\begin{aligned} v_0 &= -\frac{1}{\theta_0}\sum_1^\infty \sin\frac{n\pi}{\theta_0}\theta \sin\frac{n\pi}{\theta_0}\theta' \int \alpha e^{-\kappa\alpha^2 t} J_{\frac{n\pi}{\theta_0}}(\alpha r') H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r)\, d\alpha \quad (r>r') \\ &= -\frac{1}{\theta_0}\sum_1^\infty \sin\frac{n\pi}{\theta_0}\theta \sin\frac{n\pi}{\theta_0}\theta' \int \alpha e^{-\kappa\alpha^2 t} J_{\frac{n\pi}{\theta_0}}(\alpha r) H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r')\, d\alpha, \quad (r<r') \end{aligned}\right\} \quad (1)$$

the integrals being taken over the path (P) of Fig. 14.

This leads, as in § 88, to the solution for the source at (r', θ') in the solid bounded by $r=a$ and $\theta=0$, $\theta=\theta_0$:

$$v = -\frac{1}{\theta_0}\sum_1^\infty \sin\frac{n\pi}{\theta_0}\theta \sin\frac{n\pi}{\theta_0}\theta'$$

$$\times \int \alpha e^{-\kappa\alpha^2 t} \frac{J_{\frac{n\pi}{\theta_0}}(\alpha r')}{J_{\frac{n\pi}{\theta_0}}(\alpha a)} \left\{ H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r) J_{\frac{n\pi}{\theta_0}}(\alpha a) - H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a) J_{\frac{n\pi}{\theta_0}}(\alpha r) \right\} d\alpha, \quad (2)$$

when $r>r'$, the integral being taken over the path (P); and when $r<r'$, we interchange r and r'.

This solution can be reduced to an infinite series as before by introducing the path (Q) of Fig. 17.

Then we obtain

$$v = \frac{2i\pi}{a\theta_0}\sum_1^\infty \sum_\alpha \sin\frac{n\pi}{\theta_0}\theta \sin\frac{n\pi}{\theta_0}\theta' \alpha e^{-\kappa\alpha^2 t} \frac{J_{\frac{n\pi}{\theta_0}}(\alpha r) J_{\frac{n\pi}{\theta_0}}(\alpha r') H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a)}{J'_{\frac{n\pi}{\theta_0}}(\alpha a)}, \quad \ldots(3)$$

the summation being taken over the positive roots of $J_{\frac{n\pi}{\theta_0}}(\alpha a)=0$.

But, as in § 88, $\quad H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a) J'_{\frac{n\pi}{\theta_0}}(\alpha a) = -\frac{2i}{\pi\alpha a}.$

Thus (3) reduces to

$$v = \frac{4}{a^2\theta_0}\sum_1^\infty \sum_\alpha \sin\frac{n\pi}{\theta_0}\theta \sin\frac{n\pi}{\theta_0}\theta' e^{-\kappa\alpha^2 t} \frac{J_{\frac{n\pi}{\theta_0}}(\alpha r) J_{\frac{n\pi}{\theta_0}}(\alpha r')}{\left[J'_{\frac{n\pi}{\theta_0}}(\alpha a)\right]^2}, \quad \ldots\ldots(4)$$

which holds when $r \gtrless r'$.

The solution for the initial temperature $f(r, \theta)$ follows at once from (4) by integration, and we have, for this case,

$$v = \frac{4}{a^2\theta_0}\int_0^a \int_0^{\theta_0} r' f(r', \theta')$$

$$\times \left[\sum_1^\infty \sum_\alpha \sin\frac{n\pi}{\theta_0}\theta \sin\frac{n\pi}{\theta_0}\theta' e^{-\kappa\alpha^2 t} \frac{J_{\frac{n\pi}{\theta_0}}(\alpha r) J_{\frac{n\pi}{\theta_0}}(\alpha r')}{\left[J'_{\frac{n\pi}{\theta_0}}(\alpha a)\right]^2} \right] dr'\, d\theta'.$$

[Cf. § 62, II.]

92. The following results can be obtained by the method of the preceding section. In each case the surface of the solid is kept at temperature zero, and a unit source is placed at $t=0$ at a point of the solid.

I. *The Solid bounded internally by the Cylinder $r=a$. Line Source at (r', θ').*

Here the temperature at (r, θ) at the time t is given by

$$v=\frac{1}{4\pi}\sum_{-\infty}^{\infty}\cos n(\theta-\theta')\int_{-\infty}^{\infty}\alpha e^{-\kappa\alpha^2 t}\frac{H_n^{(1)}(\alpha r')}{H_n^{(1)}(\alpha a)}$$
$$\times\{J_n(\alpha r)H_n^{(1)}(\alpha a)-J_n(\alpha a)H_n^{(1)}(\alpha r)\}\,d\alpha,$$

when $r<r'$. We interchange r and r', when $r>r'$.

II. *The Solid bounded internally by the Cylinder $r=a$, and the Planes $\theta=0$, $\theta=\theta_0$. Line Source at (r', θ').*

In this case we are dealing with the region

$$r\geqq a,\quad 0\leqq\theta\leqq\theta_0.$$

Then we have

$$v=\frac{1}{\theta_0}\sum_{1}^{\infty}\sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'\int_{-\infty}^{\infty}\alpha e^{-\kappa\alpha^2 t}\frac{H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r')}{H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a)}$$
$$\times\left\{J_{\frac{n\pi}{\theta_0}}(\alpha r)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a)-J_{\frac{n\pi}{\theta_0}}(\alpha a)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r)\right\}d\alpha,$$

when $r<r'$. We interchange r and r', when $r>r'$.

III. *The Solid bounded by the Cylinders $r=a$ and $r=b$. Line Source at (r', θ').*

In this case we are dealing with the region

$$a\leqq r\leqq b,\quad 0\leqq\theta\leqq 2\pi.$$

Then we have

$$v=\frac{i}{2}\sum_{-\infty}^{\infty}\sum_{\alpha}\cos n(\theta-\theta')\alpha e^{-\kappa\alpha^2 t}\frac{\{J_n(\alpha a)H_n^{(1)}(\alpha r')-J_n(\alpha r')H_n^{(1)}(\alpha a)\}\times\{J_n(\alpha b)H_n^{(1)}(\alpha r)-J_n(\alpha r)H_n^{(1)}(\alpha b)\}}{\frac{d}{d\alpha}\{J_n(\alpha a)H_n^{(1)}(\alpha b)-J_n(\alpha b)H_n^{(1)}(\alpha a)\}},$$

when $r>r'$. We interchange r and r', when $r<r'$.

The summation in α is taken over the positive roots of

$$J_n(\alpha a)H_n^{(1)}(\alpha b)-J_n(\alpha b)H_n^{(1)}(\alpha a)=0.$$

The above result reduces to

$$v=\frac{\pi}{4}\sum_{-\infty}^{\infty}\sum_{\alpha}\cos n(\theta-\theta')\alpha^2 e^{-\kappa\alpha^2 t}\frac{J_n^2(\alpha a)}{J_n^2(\alpha b)-J_n^2(\alpha a)}U_n(\alpha r)U_n(\alpha r'),$$

where $U_n(\alpha r)=J_n(\alpha r)H_n^{(1)}(\alpha b)-J_n(\alpha b)H_n^{(1)}(\alpha r)$. (Cf. § 62, I.)

IV. *The Solid bounded by the Cylinders $r=a$ and $r=b$, and the Planes $\theta=0$, $\theta=\theta_0$.*

In this case we are dealing with the region

$$a\leqq r\leqq b,\quad 0\leqq\theta\leqq\theta_0.$$

Then we have

$$v=\frac{2i\pi}{\theta_0}\sum_1^\infty\sum_\alpha \sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'\alpha e^{-\kappa\alpha^2 t}\frac{\left\{J_{\frac{n\pi}{\theta_0}}(\alpha a)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r')-J_{\frac{n\pi}{\theta_0}}(\alpha r')H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a)\right\}\times\left\{J_{\frac{n\pi}{\theta_0}}(\alpha b)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r)-J_{\frac{n\pi}{\theta_0}}(\alpha r)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha b)\right\}}{\frac{d}{d\alpha}\left\{J_{\frac{n\pi}{\theta_0}}(\alpha a)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha b)-J_{\frac{n\pi}{\theta_0}}(\alpha b)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a)\right\}}$$

when $r>r'$. We interchange r and r', when $r<r'$.

The summation in α is taken over the positive roots of

$$J_{\frac{n\pi}{\theta_0}}(\alpha a)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha b)-J_{\frac{n\pi}{\theta_0}}(\alpha b)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha a)=0.$$

The above result reduces to

$$v=\frac{\pi^2}{\theta_0}\sum_1^\infty\sum_\alpha \sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'\alpha^2 e^{-\kappa\alpha^2 t}\frac{J^2_{\frac{n\pi}{\theta_0}}(\alpha a)}{J^2_{\frac{n\pi}{\theta_0}}(\alpha b)-J^2_{\frac{n\pi}{\theta_0}}(\alpha a)}U_{\frac{n\pi}{\theta_0}}(\alpha r)\,U_{\frac{n\pi}{\theta_0}}(\alpha r'),$$

where
$$U_{\frac{n\pi}{\theta_0}}(\alpha r)=J_{\frac{n\pi}{\theta_0}}(\alpha r)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha b)-J_{\frac{n\pi}{\theta_0}}(\alpha b)H^{(1)}_{\frac{n\pi}{\theta_0}}(\alpha r).$$

V. *The Wedge bounded by the Planes $z=0$, $z=h$, $\theta=0$ and $\theta=\theta_0$. Point Source at (r', θ', z').*

In this case we are dealing with the region

$$r\geqq 0,\quad 0\leqq\theta\leqq\theta_0,\quad 0\leqq z\leqq h.$$

Then we have

$$v=\frac{4}{h\theta_0}\sum_{m=1}^\infty\sum_{n=1}^\infty e^{-\kappa\frac{m^2\pi^2}{h^2}t}\sin\frac{m\pi}{h}z\sin\frac{m\pi}{h}z'\sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'$$
$$\times\int_0^\infty \alpha e^{-\kappa\alpha^2 t}J_{\frac{n\pi}{\theta_0}}(\alpha r)J_{\frac{n\pi}{\theta_0}}(\alpha r')\,d\alpha.$$

VI. *The Solid bounded by the Cylinder $r=a$, and the Planes $z=0$, $z=h$, $\theta=0$ and $\theta=\theta_0$. Point Source at (r', θ', z').*

In this case we are dealing with the region

$$0\leqq r\leqq a,\quad 0\leqq\theta\leqq\theta_0,\quad 0\leqq z\leqq h.$$

Then we have

$$v=\frac{8}{a^2h\theta_0}\sum_{m=1}^\infty\sum_{n=1}^\infty\sum_\alpha e^{-\kappa\frac{m^2\pi^2}{h^2}t}\sin\frac{m\pi}{h}z\sin\frac{m\pi}{h}z'\sin\frac{n\pi}{\theta_0}\theta\sin\frac{n\pi}{\theta_0}\theta'$$
$$\times e^{-\kappa\alpha^2 t}\frac{J_{\frac{n\pi}{\theta_0}}(\alpha r)J_{\frac{n\pi}{\theta_0}}(\alpha r')}{\left[J'_{\frac{n\pi}{\theta_0}}(\alpha a)\right]^2},$$

the summation in α being taken over the positive roots of $J_{\frac{n\pi}{\theta_0}}(\alpha a)=0$. (Cf. § 62, III.)

93. The Sphere r=a. Initial Temperature f(r, θ, φ). Surface Temperature Zero.

In this case we start with

$$v_0=\frac{1}{(2\sqrt{(\pi\kappa t)})^3}e^{-\frac{R^2}{4\kappa t}},$$

where $R^2=r^2+r'^2-2rr'\cos\gamma$, with the usual notation, γ being the angle between the radii to (r, θ, ϕ) and (r', θ', ϕ').

Then
$$v_0=\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{(2\sqrt{(\pi\kappa t)})^3}e^{\frac{rr'}{2\kappa t}\cos\gamma}.$$

But from the expansion of $e^{z\cos\theta}$ in a series of Zonal Harmonics,* we have

$$e^{\frac{rr'}{2\kappa t}\cos\gamma}=\sqrt{\left(\frac{\pi\kappa t}{rr'}\right)}\sum_0^\infty(2n+1)\frac{1}{i^{n+\frac{1}{2}}}J_{n+\frac{1}{2}}\left(\frac{irr'}{2\kappa t}\right)P_n(\cos\gamma).$$

Thus
$$v_0=\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{8\pi\kappa t\sqrt{(rr')}}\sum_0^\infty(2n+1)\frac{1}{i^{n+\frac{1}{2}}}J_{n+\frac{1}{2}}\left(\frac{irr'}{2\kappa t}\right)P_n(\cos\gamma).$$

But
$$\int_0^\infty ae^{-\kappa a^2t}J_{n+\frac{1}{2}}(ar)J_{n+\frac{1}{2}}(ar')da=\frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{2\kappa t}\frac{1}{i^{n+\frac{1}{2}}}J_{n+\frac{1}{2}}\left(\frac{irr'}{2\kappa t}\right)\dagger.$$

Hence

$$v_0=\frac{1}{4\pi\sqrt{(rr')}}\sum_0^\infty(2n+1)P_n(\cos\gamma)\int_0^\infty ae^{-\kappa a^2t}J_{n+\frac{1}{2}}(ar)J_{n+\frac{1}{2}}(ar')da. \quad\text{...(1)}$$

But, as in § 88,

$$\int_0^\infty ae^{-\kappa a^2t}J_{n+\frac{1}{2}}(ar)J_{n+\frac{1}{2}}(ar')da=-\frac{1}{2}\int ae^{-\kappa a^2t}J_{n+\frac{1}{2}}(ar')H^{(1)}_{n+\frac{1}{2}}(ar)da,$$

over the path (P) of Fig. 14, when $r>r'$; and we interchange r, r' when $r<r'$.

Therefore we have

$$v_0=-\frac{1}{8\pi\sqrt{(rr')}}\sum_0^\infty(2n+1)P_n(\cos\gamma)\int ae^{-\kappa a^2t}J_{n+\frac{1}{2}}(ar')H^{(1)}_{n+\frac{1}{2}}(ar)da, \quad (2)$$

when $r>r'$, the integral being taken over the path (P), and r, r' being interchanged when $r<r'$.

Now let

$$v_1=\frac{1}{8\pi\sqrt{(rr')}}\sum_0^\infty(2n+1)P_n(\cos\gamma)\int Aae^{-\kappa a^2t}J_{n+\frac{1}{2}}(ar)J_{n+\frac{1}{2}}(ar')da,$$

over the path (P), and choose A so that v_0+v_1 vanishes when $r=a$.

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* Cf. Heine, *loc. cit.*, Bd. I., p. 82 (14). † Cf. footnote, p. 193.

This leads to $$A=\frac{H^{(1)}_{n+\frac{1}{2}}(\alpha a)}{J_{n+\frac{1}{2}}(\alpha a)}.$$

Also we have

$$v=v_0+v_1$$
$$=-\frac{1}{8\pi\sqrt{(rr')}}\sum_0^\infty(2n+1)P_n(\cos\gamma)$$
$$\times\int\alpha e^{-\kappa\alpha^2 t}\frac{J_{n+\frac{1}{2}}(\alpha r')}{J_{n+\frac{1}{2}}(\alpha a)}\{J_{n+\frac{1}{2}}(\alpha a)H^{(1)}_{n+\frac{1}{2}}(\alpha r)-J_{n+\frac{1}{2}}(\alpha r)H^{(1)}_{n+\frac{1}{2}}(\alpha a)\}d\alpha, \quad (3)$$

when $r>r'$; while we interchange r and r' when $r<r'$.

It can be shown, as before, that this value of v satisfies all the conditions for a source at (r', θ', ϕ') at $t=0$ in the sphere.

This solution can be reduced to an infinite series by introducing the path (Q) of Fig. 17.

Then we obtain

$$v=\frac{i}{4\sqrt{(rr')}}\sum_{n=0}^\infty\sum_\alpha(2n+1)P_n(\cos\gamma)\alpha e^{-\kappa\alpha^2 t}\frac{J_{n+\frac{1}{2}}(\alpha r)J_{n+\frac{1}{2}}(\alpha r')H^{(1)}_{n+\frac{1}{2}}(\alpha a)}{aJ'_{n+\frac{1}{2}}(\alpha a)} \quad (4)$$

the summation in α being taken over the positive roots of $J_{n+\frac{1}{2}}(\alpha a)=0$.

But, as in § 88, $$H^{(1)}_{n+\frac{1}{2}}(\alpha a)J'_{n+\frac{1}{2}}(\alpha a)=-\frac{2i}{\pi\alpha a}.$$

Then from (4), we have

$$v=\frac{1}{2\pi a^2\sqrt{(rr')}}\sum_{n=0}^\infty\sum_\alpha(2n+1)P_n(\cos\gamma)e^{-\kappa\alpha^2 t}\frac{J_{n+\frac{1}{2}}(\alpha r)J_{n+\frac{1}{2}}(\alpha r')}{[J'_{n+\frac{1}{2}}(\alpha a)]^2}, \quad (5)$$

the summation in α being as above.

If the initial temperature of the sphere is $f(r, \theta, \phi)$, the temperature at (r, θ, ϕ) at the time t follows from (5) by integration and we have

$$v=\frac{1}{2\pi a^2\sqrt{r}}\int_0^a\int_0^\pi\int_0^{2\pi}f(r', \theta', \phi')\sum_{n=0}^\infty\sum_\alpha(2n+1)P_n(\cos\gamma)$$
$$\times e^{-\kappa\alpha^2 t}\frac{J_{n+\frac{1}{2}}(\alpha r)J_{n+\frac{1}{2}}(\alpha r')}{[J'_{n+\frac{1}{2}}(\alpha a)]^2}r'^{\frac{3}{2}}\sin\theta'\,dr'\,d\theta'\,d\phi'.$$

Remembering that

$$P_n(\cos\gamma)=P_n(\mu)P_n(\mu')+2\sum_{m=1}^\infty\frac{(n-m)!}{(n+m)!}(1-\mu^2)^{\frac{m}{2}}D^mP_n(\mu)$$
$$\times(1-\mu'^2)^{\frac{m}{2}}D^mP_n(\mu')\cos m(\phi-\phi'),$$

it will be seen that this solution agrees with the result obtain in § 67.

However, in the discussion in this section, we have not assume that the arbitrary function $f(r, \theta, \phi)$ can be expanded in a series terms of the form

$$(ar)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(ar)(1-\mu^2)^{\frac{m}{2}}D^m P_n(\mu)\frac{\cos}{\sin}m\phi.$$

CHAPTER XI

THE USE OF CONTOUR INTEGRALS IN THE SOLUTION OF THE EQUATION OF CONDUCTION

94. Introductory.

In the previous chapter we have obtained the Green's functions in several cases by integrating suitable solutions along a path in the plane of the complex variable. The same method can be applied in other cases, and, indeed, it is the simplest and most direct way of solving many problems of conduction. In this chapter we shall apply it to some problems already solved by elementary methods, and to others which, so far, have not been solved at all, or have only been treated by Heaviside's "operational method."* In this class may be mentioned the problem of the semi-infinite rod composed of two materials, the end kept at a constant temperature, the initial temperature of the whole being zero, and the corresponding problems for the finite rod and the sphere. The methods used in the solution of these three problems give equally satisfactory results when the surface temperature varies with the time, or radiation takes place into a medium at a constant or varying temperature.

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* Heaviside's "operational method" may be said to be simply a kind of short-hand. The formulae can be established by the use of the contour integrals employed in the following pages. And the results are confirmed in this chapter. But his work is hard to follow, and it may safely be said that he makes little attempt to justify the steps in his argument. Indeed the real justification of his method seems to depend upon some such use of contour integrals as will be found below.

Reference should be made to Heaviside's *Electromagnetic Theory*, Vol. II., Chapter V., and to his paper in *London, Proc. R. Soc.*, 52, p. 504, 1893; also to Bromwich's papers, *London, Proc. Math. Soc.* (Ser. 2), 15, p. 401, 1917; *Phil. Mag., London* (Ser. 6), 37, p. 407, 1919; and *Cambridge, Proc. Phil. Soc.*, 20, p. 411, 1921.

The method employed in this chapter was given by the author in his paper in *Phil. Mag., London* (Ser. 6), 39, p. 603, 1920. See also *Cambridge, Proc. Phil. Soc.*, 20, p. 399, 1921.

LINEAR FLOW.

95. Semi-Infinite Rod ($x > 0$). End $x=0$ kept at Constant Temperature v_0. Initial Temperature Zero.

Consider the integral $v=\dfrac{v_0}{i\pi}\displaystyle\int e^{iax}\frac{e^{-\kappa a^2 t}}{a}\,da$ ....(1)

over the standard path (P) of Fig. 14. In this path at infinity in the right the argument of a lies between 0 and $\frac{1}{4}\pi$, and on the left between $\frac{3}{4}\pi$ and π.

This value of v satisfies the equation

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}.$$

It also satisfies the conditions at $x=0$ and $t=0$.

For the initial condition put $t=0$ in (1), and we have

$$\frac{v_0}{i\pi}\int\frac{e^{iax}}{a}\,da, \text{ over the path } (P) \text{ of Fig. 14.*}$$

Consider this integral over the closed circuit of Fig. 16, consisting of the path (P) and the part of a circle, centre at the origin, lying above the path (P). There are no poles of the integrand inside this circuit, and therefore the integral over the whole vanishes. But when the radius of the circle tends to infinity, the integral over the circular arc vanishes, x being positive. It follows that the integral over the complete path (P) vanishes when x is positive.

For the boundary condition put $x=0$ in (1), and we have

$$\frac{v_0}{i\pi}\int\frac{e^{-\kappa a^2 t}}{a}\,da, \text{ over the path } (P).$$

This is equal to

$$\frac{v_0}{2i\pi}\int\frac{e^{-\kappa a^2 t}}{a}\,da, \text{ over the path } (Q) \text{ of Fig 17,}$$

since the integrand is an odd function of a, and the integrals over

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* For a more exact discussion, we should consider $\underset{t\to 0}{\mathrm{Lt}}\,(v)$, when x is a given positive number, and $\underset{x\to 0}{\mathrm{Lt}}\,(v)$, when t is a given positive number. It is not difficult to show that the value of v given in (1) is a continuous function of t, when $t \geqq 0$, x being a given positive number, and a continuous function of x, when $x \geqq 0$, t being a given positive number.

The same remark applies to the discussion of the boundary and initial conditions throughout this chapter.

the circular arcs at infinity (dotted in the diagram) vanish, when t is positive.

It follows by Cauchy's Theorem that, when $x=0$, v is equal to v_0.

Thus the value of v given in (1) satisfies all the conditions of our problem.

But the path (P) can be replaced in (1) by the straight path of Fig. 21, with the semi-circle enclosing the origin; and, by letting

FIG. 21.

the radius of this circle tend to zero, we obtain our solution in the form

$$v=1-\frac{2}{\pi}\int_0^\infty e^{-\kappa\alpha^2 t}\frac{\sin\alpha x}{\alpha}\,d\alpha. \qquad \text{.................(2)}$$

Now it is known that

$$\int_0^\infty e^{-a^2x^2}\cos bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}},$$

and that we can integrate under the sign of integration. (Cf. *F.S.*, p. 195, Ex. 13.)

Thus $$\int_0^\infty e^{-a^2x^2}\frac{\sin bx}{x}\,dx=\frac{\sqrt{\pi}}{2a}\int_0^b e^{-\frac{b^2}{4a^2}}\,db.$$

It follows from (2) that

$$v=1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{(\kappa t)}}} e^{-\xi^2}\,d\xi. \quad \text{(Cf. p. 35, Ex. 1.)}$$

96. Semi-Infinite Rod ($x>0$). End $x=0$ kept at Temperature $A\cos\omega t$. Initial Temperature Zero.

Consider the integral

$$v=\frac{A}{i\pi}\int e^{i\alpha x}e^{-\kappa\alpha^2 t}\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2} \qquad \text{......................(1)}$$

over the standard path (P) of Fig. 14.*

This value of v satisfies the equation $\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}$, and the same argument as in § 95 shows that it also satisfies the condition at $t=0$.

For the condition at the end of the rod, we take the integral

$$\frac{A}{i\pi}\int e^{-\kappa\alpha^2 t}\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2} \text{ over the path } (P).$$

*In this case the path (P) is to lie above the points $\sqrt{\left(\frac{\omega}{\kappa}\right)}e^{\frac{1}{4}i\pi}$ and $\sqrt{\left(\frac{\omega}{\kappa}\right)}e^{\frac{3}{4}i\pi}$.

This is equal to

$$\frac{a}{2i\pi}\int e^{-\kappa\alpha^2 t}\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2} \text{ over the path } (Q) \text{ of Fig. 17.}$$

and the poles of the integrand are at $\sqrt{\left(\frac{\omega}{\kappa}\right)}e^{\pm\frac{1}{4}i\pi}$, $\sqrt{\left(\frac{\omega}{\kappa}\right)}e^{\pm\frac{3}{4}i\pi}$.

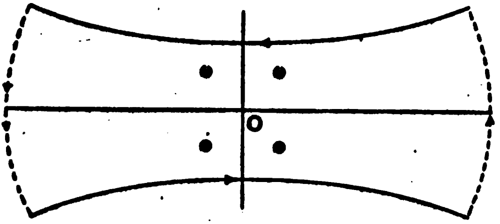

FIG. 22.

It follows by Cauchy's Theorem that, when $x=0$, we have $v=a\cos\omega t$.

Thus the value of v given in (1) satisfies all the conditions of our problem.

But the path (P) can be replaced in (1) by the straight path of Fig. 23 from right to left, with the small circles enclosing the points

$$\sqrt{\left(\frac{\omega}{\kappa}\right)}e^{\frac{1}{4}i\pi} \quad\text{and}\quad \sqrt{\left(\frac{\omega}{\kappa}\right)}e^{\frac{3}{4}i\pi}.$$

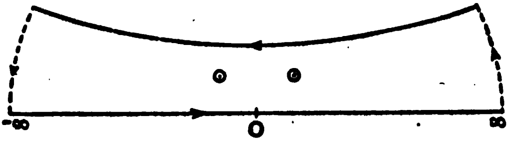

FIG. 23.

We thus obtain our solution in the form

$$v=a\,e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x\right)-\frac{2a}{\pi}\int_0^\infty e^{-\kappa\alpha^2 t}\sin\alpha x\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2}. \quad\ldots(2)$$

It can be shown that

$$\int_0^\infty e^{-\lambda t}\frac{\lambda}{\lambda^2+\omega^2}\sin\sqrt{\left(\frac{\lambda}{\kappa}\right)}x\,d\lambda=2\sqrt{\pi}\int_0^{\frac{x}{2\sqrt{(\kappa t)}}}e^{-\mu^2}\cos\omega\left(t-\frac{x^2}{4\kappa\mu^2}\right)d\mu,$$

and from this equality the solution in (2) can be reduced to

$$v=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{(\kappa t)}}}^{\infty}e^{-\mu^2}\cos\omega\left(t-\frac{x^2}{4\kappa\mu^2}\right)d\mu,$$

as found in § 23.

If the temperature of the end $x=0$ is kept at $a\sin\omega t$, we start with the solution

$$v=-\frac{a\omega}{i\kappa\pi}\int e^{i\alpha x}e^{-\kappa\alpha^2 t}\frac{\alpha\,d\alpha}{\alpha^4+\omega^2/\kappa^2},$$

and the real form of the answer is

$$v=a\,e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\sin\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x\right)+\frac{2a\omega}{\kappa\pi}\int_0^\infty e^{-\kappa\alpha^2 t}\sin\alpha x\frac{\alpha\,d\alpha}{\alpha^4+\omega^2/\kappa^2}.$$

97. Semi-Infinite Rod ($x>0$). Radiation at $x=0$ into a Medium at Constant Temperature v_0. Initial Temperature Zero.

Here we take
$$v=\frac{hv_0}{i\pi}\int e^{i\alpha x}\frac{e^{-\kappa\alpha^2 t}}{h-i\alpha}\frac{d\alpha}{\alpha}\quad\text{..........................(1)}$$

over the standard path (P) of Fig. 14.

It follows as in § 95 that all the conditions of the problem are satisfied by this value of v, the condition at $x=0$ now being

$$-\frac{\partial v}{\partial x}+h(v-v_0)=0.$$

We may replace the path (P) in (1), as in § 95, by the straight path of Fig. 21, and by letting the radius of the semi-circle tend to zero we obtain our solution in the form

$$v=v_0-\frac{2hv_0}{\pi}\int_0^\infty e^{-\kappa\alpha^2 t}\frac{h\sin\alpha x+\alpha\cos\alpha x}{h^2+\alpha^2}\frac{d\alpha}{\alpha}.\quad\text{..........(2)}$$

By using the integral

$$\int_x^\infty e^{-hx}\sin\alpha x\,dx=\frac{h\sin\alpha x+\alpha\cos\alpha x}{h^2+\alpha^2}e^{-hx},$$

the solution in (2) can be reduced to

$$v=\frac{2hv_0}{\sqrt{\pi}}\int_0^\infty e^{-h\mu}\,d\mu\int_{\frac{x+\mu}{2\sqrt{(\kappa t)}}}^\infty e^{-\eta^2}\,d\eta,$$

as obtained in § 25.

The gradient $\frac{\partial v}{\partial x}$ at $x=0$ follows directly from the value of v in (1).

For we have
$$\left(\frac{\partial v}{\partial x}\right)_{x=0}=hv_0\int\frac{e^{-\kappa\alpha^2 t}}{h-i\alpha}\,d\alpha,\ \text{over the path }(P),$$

$$=-\frac{2h^2v_0}{\pi}\int_0^\infty\frac{e^{-\kappa\alpha^2 t}}{h^2+\alpha^2}\,d\alpha,$$

and from this we obtain the approximate value

$$\frac{\partial v}{\partial x}=-\frac{v_0}{\sqrt{(\pi\kappa t)}}\left\{1-\frac{1}{2h^2\kappa t}\right\}$$

as on p. 52.

98. Semi-Infinite Rod ($x>0$). Radiation at $x=0$ into a Medium at Temperature $a\cos\omega t$. Initial Temperature Zero.

Here we take
$$v=\frac{ha}{i\pi}\int e^{i\alpha x}\frac{e^{-\kappa\alpha^2 t}}{h-i\alpha}\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2}\quad\text{.....................(1)}$$
over the standard path (P) of Fig. 14.*

It follows as before that all the conditions of the problem are satisfied by this value of v, the condition at $x=0$ now being
$$-\frac{\partial v}{\partial x}+h(v-a\cos\omega t)=0.$$

The solution given in (1) reduces to
$$v=\frac{ha}{\sqrt{\left\{\left(h+\sqrt{\left(\frac{\omega}{2\kappa}\right)}\right)^2+\frac{\omega}{2\kappa}\right\}}}e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\cos\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x-\gamma\right)$$
$$-\frac{2ha}{\pi}\int_0^\infty e^{-\kappa\alpha^2 t}\frac{h\sin\alpha x+\alpha\cos\alpha x}{h^2+\alpha^2}\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2},$$
where
$$\gamma=\tan^{-1}\frac{\sqrt{(\omega/2\kappa)}}{h+\sqrt{(\omega/2\kappa)}}.$$

For radiation at $x=0$ into a medium at $a\sin\omega t$, we start with the integral
$$v=-\frac{ha\omega}{i\kappa\pi}\int e^{i\alpha x}\frac{e^{-\kappa\alpha^2 t}}{h-i\alpha}\frac{\alpha\,d\alpha}{\alpha^4+\omega^2/\kappa^2}$$
over the path (P).

The solution reduces to
$$v=\frac{ha}{\sqrt{\left\{\left(h+\sqrt{\left(\frac{\omega}{2\kappa}\right)}\right)^2+\frac{\omega}{2\kappa}\right\}}}e^{-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x}\sin\left(\omega t-\sqrt{\left(\frac{\omega}{2\kappa}\right)}x-\gamma\right)$$
$$+\frac{2ha\omega}{\kappa\pi}\int e^{-\kappa\alpha^2 t}\frac{h\sin\alpha x+\alpha\cos\alpha x}{h^2+\alpha^2}\frac{\alpha\,d\alpha}{\alpha^4+\omega^2/\kappa^2},$$
where γ has the same value as above.

99. Semi-Infinite Rod ($x>-a$). From $x=-a$ to $x=0$ of one Material: from $x=0$ to Infinity of another. End $x=-a$ kept at Constant Temperature v_0. Initial Temperature Zero.

With the usual notation, let v_1, K_1, c_1, ρ_1 refer to the interval $-a<x<0$, and v_2, K_2, c_2, ρ_2 refer to $x>0$.

Also let
$$\kappa_1=K_1/c_1\rho_1\quad\text{and}\quad\kappa_2=K_2/c_2\rho_2.$$

* The footnote on p. 203 applies here also.

Then the equations to be solved are as follows:

(1) $\dfrac{\partial v_1}{\partial t}=\kappa_1\dfrac{\partial^2 v_1}{\partial x^2}$, $-a<x<0$: $\dfrac{\partial v_2}{\partial t}=\kappa_2\dfrac{\partial^2 v_2}{\partial x^2}$, $x>0$. ....(1′)

(2) $v_1=v_0$, when $x=-a$.

(3) $v_1=0$, when $t=0$, $-a<x<0$: $v_2=0$, when $t=0$, $x>0$. (3′)

(4) $v_1=v_2$, when $x=0$,

(5) $K_1\dfrac{\partial v_1}{\partial x}=K_2\dfrac{\partial v_2}{\partial x}$ when $x=0$.

It is clear that
$$v_1=(A_1e^{i\alpha x}+B_1e^{-i\alpha x})e^{-\kappa_1\alpha^2 t},$$
$$v_2=A_2e^{i\mu\alpha x-\kappa_1\alpha^2 t},$$
where $\mu=\sqrt{(\kappa_1/\kappa_2)}$, satisfy (1) and (1′).

They also satisfy (4) and (5), provided that
$$\left.\begin{aligned}A_1+B_1&=A_2,\\ K_1(A_1-B_1)&=\mu K_2A_2.\end{aligned}\right\}$$

Thus $A_1=\tfrac{1}{2}(1+\sigma)A_2$, $B_1=\tfrac{1}{2}(1-\sigma)A_2$,

where $\sigma=\mu K_2/K_1=\sqrt{(K_2c_2\rho_2/K_1c_1\rho_1)}$.

Introducing the path (P) of Fig. 14 and choosing a suitable value for A_2 (this value is indicated by (2)), we are led to the solutions:
$$v_1=\frac{v_0}{i\pi}\int\frac{(1+\sigma)e^{i\alpha x}+(1-\sigma)e^{-i\alpha x}}{(1+\sigma)e^{-i\alpha a}+(1-\sigma)e^{i\alpha a}}\frac{e^{-\kappa_1\alpha^2 t}}{\alpha}d\alpha \quad ....(6)$$
and
$$v_2=\frac{2v_0}{i\pi}\int\frac{e^{i\mu\alpha x}}{(1+\sigma)e^{-i\alpha a}+(1-\sigma)e^{i\alpha a}}\frac{e^{-\kappa_1\alpha^2 t}}{\alpha}d\alpha, \quad ....(7)$$
the integrals in both cases being taken over the standard path (P).

The expressions in (6) and (7) satisfy the differential equations (1) and (1′), and the conditions at $x=0$ given in (4) and (5).

They also satisfy the remaining conditions (2), (3) and (3′).

For $\dfrac{v_0}{i\pi}\displaystyle\int\frac{e^{-\kappa_1\alpha^2 t}}{\alpha}d\alpha$ over the path (P)

is equal, as before, to v_0.

Also the roots of the equation
$$(1+\sigma)e^{-i\alpha a}+(1-\sigma)e^{i\alpha a}=0$$
are given by
$$\alpha=\frac{n\pi}{a}\qquad-\frac{i}{2a}\log\left(\frac{\sigma+1}{\sigma-1}\right),\ \text{when } \sigma>1,$$
and
$$\alpha=\frac{2n+1}{2a}\pi-\frac{i}{2a}\log\left(\frac{1+\sigma}{1-\sigma}\right),\ \text{when } 0<\sigma<1.$$

We may therefore use Fig. 16 as before, and we see that v_1 and v_2 vanish, when $t=0$.

We shall now simplify the solution of our problem we have obtained in the form of a contour integral.

I. $\sigma>1$. Put $\sigma=\coth\theta$.

Then $$\frac{(1+\sigma)e^{i\alpha a}+(1-\sigma)e^{-i\alpha a}}{(1+\sigma)e^{-i\alpha a}+(1-\sigma)e^{i\alpha a}}=-\frac{\sin(\alpha x-i\theta)}{\sin(\alpha a+i\theta)}.$$

Thus $$v_1=-\frac{v_0}{i\pi}\int\frac{\sin(\alpha x-i\theta)}{\sin(\alpha a+i\theta)}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha, \text{ over the path } (P),$$

$$=v_0+\frac{v_0}{i\pi}\int_0^\infty\left\{\frac{\sin(\alpha x-i\theta)}{\sin(\alpha a+i\theta)}-\frac{\sin(\alpha x+i\theta)}{\sin(\alpha a-i\theta)}\right\}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha$$

$$=v_0-\frac{2v_0\sinh 2\theta}{\pi}\int_0^\infty\frac{\sin\alpha(x+a)}{\cosh 2\theta-\cos 2\alpha a}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha. \quad\text{.........(8)}$$

Similarly,

$$v_2=\frac{v_0\sinh\theta}{\pi}\int\frac{e^{i\mu\alpha x}}{\sin(\alpha a+i\theta)}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha, \text{ over the path } (P),$$

$$=v_0-\frac{v_0\sinh 2\theta}{\pi}\int_0^\infty\frac{(1+\tanh\theta)\sin\alpha(\mu x+a)-(1-\tanh\theta)\sin\alpha(\mu x-a)}{\cosh 2\theta-\cos 2\alpha a}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha. \quad\text{...(9)}$$

II. $\sigma=1$.

Then $$\left.\begin{aligned}v_1&=\frac{v_0}{i\pi}\int e^{i\alpha(x+a)}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha,\\ v_2&=\frac{v_0}{i\pi}\int e^{i\alpha(\mu x+a)}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha,\end{aligned}\right\}\text{ over the path } (P).$$

These reduce to

$$v_1=v_0-\frac{2v_0}{\pi}\int_0^\infty e^{-\kappa_1\alpha^2t}\frac{\sin\alpha(x+a)}{\alpha}d\alpha$$

$$=\frac{2v_0}{\sqrt{\pi}}\int_{\frac{x+a}{2\sqrt{(\kappa t)}}}^\infty e^{-\eta^2}d\eta^* \quad\text{..............................(10)}$$

and $$v_2=\frac{2v_0}{\sqrt{\pi}}\int_{\frac{\mu x+a}{2\sqrt{(\kappa t)}}}^\infty e^{-\eta^2}d\eta. \quad\text{..............................(11)}$$

III. $0<\sigma<1$. Put $\sigma=\tanh\theta$.

Then $$\frac{(1+\sigma)e^{i\alpha x}+(1-\sigma)e^{-i\alpha x}}{(1+\sigma)e^{-i\alpha a}+(1-\sigma)e^{i\alpha a}}=\frac{\cos(\alpha x-i\theta)}{\cos(\alpha a+i\theta)}.$$

---

* Using the integral $\int_0^\infty e^{-a^2x^2}\frac{\sin bx}{x}dx=\frac{\sqrt{\pi}}{2a}\int_0^b e^{-\frac{\xi^2}{4a^2}}d\xi$, as in § 95.

Also our solutions (6) and (7) reduce to

$$v_1=v_0-\frac{2v_0\sinh 2\theta}{\pi}\int_0^\infty \frac{\sin\alpha(x+a)}{\cosh 2\theta+\cos 2a\alpha}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha \quad\text{.................(12)}$$

and

$$v_2=v_0-\frac{2v_0\cosh^2\theta}{\pi}\int_0^\infty \frac{\begin{array}{c}(1+\tanh\theta)\sin\alpha(\mu x+a)\\ +(1-\tanh\theta)\sin\alpha(\mu x-a)\end{array}}{\cosh 2\theta+\cos 2a\alpha}\frac{e^{-\kappa_1\alpha^2t}}{\alpha}d\alpha. \quad\text{........(13)}$$

This problem was discussed by Heaviside (*loc. cit.*, p. 16), the gradient at $x=-a$ being required for the question of the Age of the Earth. This gradient follows at once from our solutions (6) and (7).

We have $\left(\frac{\partial v_1}{\partial x}\right)_{x=-a}=\frac{v_0}{\pi}\int\frac{(1+\sigma)e^{-ia\alpha}-(1-\sigma)e^{ia\alpha}}{(1+\sigma)e^{-ia\alpha}+(1-\sigma)e^{ia\alpha}}e^{-\kappa_1\alpha^2t}d\alpha,$

over the path (P).

Put $k=\dfrac{\sigma-1}{\sigma+1}$.

Then $\left(\frac{\partial v_1}{\partial x}\right)_{x=-a}=\frac{v_0}{\pi}\int\frac{1+k\,e^{2ia\alpha}}{1-k\,e^{2ia\alpha}}e^{-\kappa_1\alpha^2t}d\alpha$, over the path (P),

$$=\frac{v_0}{\pi}\int\left(1+2\sum_1^\infty k^n e^{2nia\alpha}\right)e^{-\kappa_1\alpha^2t}d\alpha$$

$$=-\frac{2v_0}{\pi}\int_0^\infty\left(1+2\sum_1^\infty k^n\cos 2na\alpha\right)e^{-\kappa_1\alpha^2t}d\alpha$$

$$=-\frac{v_0}{\sqrt{(\pi\kappa_1 t)}}\left(1+2\sum_1^\infty k^n e^{-\frac{n^2a^2}{\kappa_1 t}}\right).$$

Hence for large values of t we have approximately

$$\left(\frac{\partial v_1}{\partial x}\right)_{x=-a}=-\frac{v_0}{\sqrt{(\pi\kappa_1 t)}}\frac{1+k}{1-k}$$

$$=-\frac{v_0}{\sqrt{(\pi\kappa_1 t)}}\left(\frac{K_2c_2\rho_2}{K_1c_1\rho_1}\right)^{\frac{1}{2}}.$$

When the surface is kept at zero and the initial temperature of the whole solid is v_0, it is clear that the temperature gradient, when $x=-a$, will be minus the above.

But the gradient in Kelvin's classical treatment of the Age of the Earth (§28 above) was found to be $v_0/(\pi\kappa_1 t)$.

This modification of the problem, allowing for greater conductivity and capacity for heat in the interior than in the outer skin, makes the interval required for subsidence to the same gradient $(K_2c_2\rho_2)/(K_1c_1\rho_1)$ times as great as before.

With the data adopted by Perry and Heaviside,* and the above notation, $(K_2c_2\rho_2)/(K_1c_1\rho_1)$ is nearly 450. Thus Kelvin's estimate of 10^8 years would be increased to 45×10^9 years.

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* Cf. p. 221 and the footnote on p. 60.

100. Rod of Length l. The Ends $x=0$ and $x=l$ kept at Temperatures Zero and v_0, respectively. Initial Temperature Zero.

It is clear from the argument of § 95 that

$$v=\frac{v_0}{i\pi}\int\frac{\sin ax}{\sin al}\frac{e^{-\kappa a^2 t}}{a}da, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

over the standard path (P) of Fig. 14 satisfies all the conditions of our problem.

From this solution we obtain, as before,

$$v=\frac{v_0}{2i\pi}\int\frac{\sin ax}{\sin al}\frac{e^{-\kappa a^2 t}}{a}da, \text{ over the path } (Q) \text{ of Fig. 17.}$$

And finally, by Cauchy's Theorem, we have

$$v=v_0\left[\frac{x}{l}+\frac{2}{\pi}\sum_1^\infty\frac{(-1)^n}{n}\sin\frac{n\pi}{l}x e^{-\kappa\frac{n^2\pi^2}{l^2}t}\right], \quad \ldots\ldots\ldots\ldots(2)$$

since the poles of the integrand are at 0, $\pm\frac{\pi}{l}$, $\pm\frac{2\pi}{l}$, etc.

Another form of the solution can be obtained as follows:

We have, from (1),

$$v=\frac{v_0}{i\pi}\int e^{ia(l-x)}\frac{1-e^{2iax}}{1-e^{2ial}}\frac{e^{-\kappa a^2 t}}{a}da, \text{ over the path } (P),$$

$$=\frac{v_0}{i\pi}\sum_1^\infty\int\{e^{ia((2n-1)l-x)}-e^{ia((2n-1)l+x)}\}\frac{e^{-\kappa a^2 t}}{a}da$$

$$=\frac{2v_0}{\pi}\sum_1^\infty\int_0^\infty\{\sin a((2n-1)l+x)-\sin a((2n-1)l-x)\}\frac{e^{-\kappa a^2 t}}{a}da$$

$$=\frac{v_0}{\sqrt{(\pi\kappa t)}}\sum_1^\infty\left\{\int_0^{(2n-1)l+x}e^{-\frac{\xi^2}{4\kappa t}}d\xi-\int_0^{(2n-1)l-x}e^{-\frac{\xi^2}{4\kappa t}}d\xi\right\}^*$$

$$=\frac{2v_0}{\sqrt{\pi}}\sum_1^\infty\int_{\frac{(2n-1)l-x}{2\sqrt{(\kappa t)}}}^{\frac{(2n+1)l+x}{2\sqrt{(\kappa t)}}}e^{-\xi^2}d\xi. \quad \text{(Cf. § 75, III.)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

A similar treatment of (1) leads to two expressions for $\frac{\partial v}{\partial x}$.

For example, we have

$$\left(\frac{\partial v}{\partial x}\right)_{x=0}=\frac{v_0}{i\pi}\int\frac{e^{-\kappa a^2 t}}{\sin al}da, \text{ over the path } (P),$$

$$=\frac{v_0}{l}\left[1+2\Sigma(-1)^n e^{-\kappa\frac{n^2\pi^2}{l^2}t}\right]. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

---

* The footnote on p. 206 applies here also.

And
$$\left(\frac{\partial v}{\partial x}\right)_{x=0}=\frac{2v_0}{\pi}\int\frac{e^{i\alpha l}}{1-e^{2i\alpha l}}e^{-\kappa\alpha^2 t}\,d\alpha$$
$$=\frac{4v_0}{\pi}\sum_1^\infty\int_0^\infty\cos(2n-1)\alpha l\,e^{-\kappa\alpha^2 t}\,d\alpha$$
$$=\frac{2v_0}{\sqrt{(\pi\kappa t)}}\sum_1^\infty e^{-\frac{(2n-1)^2 l^2}{4\kappa t}}. \qquad\ldots\ldots\ldots(5)$$

101. Rod of Length l. The Ends $x=0$ and $x=l$ kept at Temperatures v_0 and Zero, respectively. Initial Temperature Zero.

In this case the solution is obviously
$$v=\frac{v_0}{i\pi}\int\frac{\sin\alpha(l-x)}{\sin\alpha l}\,\frac{e^{-\kappa\alpha^2 t}}{\alpha}\,d\alpha, \qquad\ldots\ldots\ldots(1)$$
over the path (P) of Fig. 14.

And this leads, as before, to
$$v=v_0\left[1-\frac{x}{l}-\frac{2}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{n\pi}{l}x\,e^{-\kappa\frac{n^2\pi^2}{l^2}t}\right]. \qquad\ldots\ldots\ldots(2)$$

The remarks as to a second form of the solution, and the gradient $\frac{\partial v}{\partial x}$, are equally applicable to this case, which, of course, could be deduced from the preceding by a change of origin.

102. Rod of Length l. The Ends $x=0$ and $x=l$ kept at Temperature Zero and Ct, respectively. Initial Temperature Zero.

In this case it is clear from the argument of § 95 that all the conditions of the problem are satisfied by
$$v=-\frac{C}{i\kappa\pi}\int\frac{\sin\alpha x}{\sin\alpha l}\,\frac{e^{-\kappa\alpha^2 t}}{\alpha^3}\,d\alpha, \qquad\ldots\ldots\ldots(1)$$
over the path (P) of Fig. 14.

Again we may write (1) in the form
$$v=-\frac{C}{2i\kappa\pi}\int\frac{\sin\alpha x}{\sin\alpha l}\,\frac{e^{-\kappa\alpha^2 t}}{\alpha^3}\,d\alpha,$$
over the path (Q) of Fig. 17.

And, finally, by Cauchy's Theorem, we have
$$v=\frac{Cx}{\kappa l}\left(\kappa t-\frac{l^2-x^2}{3!}\right)+\frac{Cl^2}{\kappa\pi^3}\sum_1^\infty\frac{(-1)^{n-1}}{n^3}\sin\frac{n\pi}{l}x\,e^{-\kappa\frac{n^2\pi^2}{l^2}t}. \quad\ldots(2)$$

When $x=l$ is kept at temperature Ct^2, Ct^3, etc., the results can be obtained in the same way.

103. Rod of Length l. The Ends $x=0$ and $x=l$ kept at Temperatures Zero and $a\cos\omega t$, respectively. Initial Temperature Zero.

In this case it is clear that all the conditions of the problem are satisfied by

$$v=\frac{a}{i\pi}\int\frac{\sin\alpha x}{\sin\alpha l}e^{-\kappa\alpha^2 t}\frac{\alpha^3}{\alpha^4+\omega^2/\kappa^2}d\alpha, \quad\text{.................(1)}$$

over the path (P) of Fig. 14.*

We may write (1) in the form

$$v=\frac{a}{2i\pi}\int\frac{\sin\alpha x}{\sin\alpha l}e^{-\kappa\alpha^2 t}\frac{\alpha^3\,d\alpha}{\alpha^4+\omega^2/\kappa^2},$$

over the path (Q) of Fig. 17.

And, finally, by Cauchy's Theorem, we have

$$v=\frac{a}{2}\left[\frac{\sin\sqrt{\left(\frac{\omega}{2\kappa}\right)}(1+i)x}{\sin\sqrt{\left(\frac{\omega}{2\kappa}\right)}(1+i)\,l}e^{-i\omega t}+\frac{\sin\sqrt{\left(\frac{\omega}{2\kappa}\right)}(1-i)x}{\sin\sqrt{\left(\frac{\omega}{2\kappa}\right)}(1-i\ l}e^{i\omega t}\right]$$

$$+\frac{2a}{l}\sum_1^\infty(-1)^n\sin\frac{n\pi}{l}x\frac{n^3\pi^3/l^3}{n^4\pi^4/l^4+\omega^2/\kappa^2}e^{-\kappa\frac{n^2\pi^2}{l^2}t}$$

$$=\frac{a}{\cosh 2\mu l-\cos 2\mu l}[\{\cos\mu(x-l)\cosh\mu(x+l)$$
$$-\cos\mu(x+l)\cosh\mu(x-l)\}\cos\omega t$$
$$-\{\sin\mu(x-l)\sinh\mu(x+l)$$
$$-\sin\mu(x+l)\sinh\mu(x-l)\}\sin\omega t]$$

$$+\frac{2a}{l}\sum_1^\infty(-1)^n\sin\frac{n\pi}{l}x\frac{n^3\pi^3/l^3}{n^4\pi^4/l^4+\omega^2/\kappa^2}e^{-\kappa\frac{n^2\pi^2}{l^2}t},$$

where $\mu=\sqrt{(\omega/2\kappa)}$.

Similar results may be obtained for the case when $x=l$ is kept at $a\sin\omega t$, and when the temperatures at $x=0$ and $x=l$ are interchanged.

104. Rod of Length l. The End $x=0$ kept at Zero. Radiation at $x=l$ into a Medium at Constant Temperature v_0. Initial Temperature Zero.

Here we have to solve the equations:

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2},\quad 0<x<l. \quad\text{.......................................(1)}$$

$$v=0,\qquad\text{when } t=0, \quad\text{.................................(2)}$$

$$\frac{\partial v}{\partial x}+h(v-v_0)=0,\ \text{when } x=l, \quad\text{.................................(3)}$$

$$v=0,\qquad\text{when } x=0. \quad\text{.................................(4)}$$

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* The footnote on p. 203 applies here also.

Starting from the solution $A \sin \alpha x e^{-\kappa\alpha^2 t}$ of (1), which vanishes when $x=0$, we are led by (3) to choose A so that

$$\alpha A \cos \alpha l + h(A \sin \alpha l - v_0) = 0,$$

i.e.
$$A = \frac{hv_0}{\alpha \cos \alpha l + h \sin \alpha l}.$$

Introducing the path (P) of Fig. 14, we obtain the solution of our problem in the form

$$v = \frac{hv_0}{i\pi}\int \frac{\sin \alpha x}{\alpha \cos \alpha l + h \sin \alpha l}\frac{e^{-\kappa\alpha^2 t}}{\alpha}\,d\alpha, \quad \text{.................(5)}$$

the integral being taken over the path (P).

It is easy, as in § 36, to show that the roots of the equation

$$\alpha \cos \alpha l + h \sin \alpha l = 0 \quad \text{..........................(6)}$$

are infinite in number, all real, and not repeated. They are symmetrical about the origin, and may be denoted by 0, $\pm\alpha_1$, $\pm\alpha_2$, etc.

The solution given in (5) reduces, by using the path (Q) of Fig. 17 and Cauchy's Theorem, to

$$v = hv_0\left\{\frac{x}{1+hl} + 2\sum_1^\infty (-1)^n \frac{\sqrt{(\alpha_n^2+h^2)}}{h(1+hl)+l\alpha_n^2}\,\frac{\sin \alpha_n x}{\alpha_n} e^{-\kappa\alpha_n^2 t}\right\}, \quad \text{...(7)}$$

the summation being taken over the positive roots of (6).

If radiation takes place at $x=0$ and $x=l$ into media at constant temperatures v_1 and v_2, respectively, we are led to the solution in the form

$$v = \frac{1}{i\pi}\int (A \sin \alpha x + B \cos \alpha x)\frac{e^{-\kappa\alpha^2 t}}{\alpha}\,d\alpha,$$

over the path (P), where A and B are determined by the equations

$$\left.\begin{aligned} -A\alpha + hB &= hv_1, \\ A[\alpha \cos \alpha l + h \sin \alpha l] + B[-\alpha \sin \alpha l + h \cos \alpha l] &= hv_2. \end{aligned}\right\}$$

The case of radiation into a medium at Ct, Ct^2, etc., or $a \cos \omega t$, $a \sin \omega t$, etc., can be treated in the same way.

For example, if radiation takes place at $x=l$ into a medium at temperature Ct, and the end $x=0$ is kept at zero, our solution is given by

$$v = -\frac{hC}{i\kappa\pi}\int \frac{\sin \alpha x}{\alpha \cos \alpha l + h \sin \alpha l}\frac{e^{-\kappa\alpha^2 t}}{\alpha^3}\,d\alpha,$$

over the path (P).

In each case we obtain the value of v in the form of an infinite series by taking the path (Q) and considering the poles of the integrand.

105. Rod of Length b composed of two different Materials. The Ends x=0 and x=b kept at Zero and Constant Temperature v_0, respectively. Initial Temperature Zero.

As in § 99, we let v_1, K_1, c_1, ρ_1 refer to the first part of the rod

$(0<x<a)$, and v_2, K_2, c_2, ρ_2 to the second $(a<x<b)$. Also we take

$$\kappa_1=K_1/c_1\rho_1 \quad \text{and} \quad \kappa_2=K_2/c_2\rho_2.$$

Then the equations to be solved are as follows:

$$(1)\ \frac{\partial v_1}{\partial t}=\kappa_1\frac{\partial^2 v_1}{\partial x^2},\quad 0<x<a: \qquad \frac{\partial v_2}{\partial t}=\kappa_2\frac{\partial^2 v_2}{\partial x^2},\quad a<x<b. \quad \ldots(1')$$

$$(2)\ v_1=0, \text{ when } x=0: \qquad v_2=v_0, \text{ when } x=b, \quad \ldots\ldots\ldots(2')$$

$$(3)\ v_1=0, \text{ when } t=0, 0<x<a: \quad v_2=0, \text{ when } t=0, a<x<b. \ (3')$$

$$(4)\ v_1=v_2, \text{ when } x=a.$$

$$(5)\ K_1\frac{\partial v_1}{\partial x}=K_2\frac{\partial v_2}{\partial x}, \text{ when } x=a.$$

It is clear that

$$v_1=A_1 \sin ax\, e^{-\kappa_1 a^2 t},$$

$$v_2=\{A_2 \sin \mu a(x-a)+B_2 \sin \mu a(b-x)\}e^{-\kappa_1 a^2 t},$$

where $\mu=\sqrt{(\kappa_1/\kappa_2)}$, satisfy (1) and (1′).

They also satisfy (4) and (5), provided that

$$\left.\begin{aligned} A_1 \sin aa&=B_2 \sin \mu a(b-a),\\ K_1A_1 \cos aa&=K_2\mu(A_2-B_2 \cos \mu a(b-a)).\end{aligned}\right\}$$

Thus we take

$$A_2=(\sigma \cos aa+\sin aa \cot \mu a(b-a))A_1,$$

$$B_2=\frac{\sin aa}{\sin \mu a(b-a)}A_1,$$

where $$\sigma=K_1/K_2\mu=\sqrt{(K_1c_1\rho_1/K_2c_2\rho_2)}.$$

Introducing the path (P) of Fig. 14, and choosing a suitable value for A_1, we are led to the solutions:

$$v_1=\frac{v_0}{i\pi}\int\frac{\sin aa}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}da, \quad \ldots\ldots\ldots(6)$$

$$v_2=\frac{v_0}{i\pi}\int\left\{\frac{\sin \mu a(x-a)}{\sin \mu a(b-a)}+\frac{\sin aa}{F(a)}\frac{\sin \mu a(b-x)}{\sin \mu a(b-a)}\right\}\frac{e^{-\kappa_1 a^2 t}}{a}da, \quad \ldots(7)$$

where $$F(a)=\sigma \cos aa \sin \mu a(b-a)+\sin aa \cos \mu a(b-a),$$

the integrals being taken over the path (P).

The value of v_2 given in (7) reduces to

$$v_2=\frac{v_0}{i\pi}\int\frac{\sigma \cos aa \sin \mu a(x-a)+\sin aa \cos \mu a(x-a)}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}da, \quad (8)$$

over the path (P).

The expressions in (6) and (8) satisfy the differential equations (1) and (1′), and the conditions at $x=0$ and $x=a$ given by (2), (4) and (5).

Further, putting $x=b$ in (8), we have

$$\frac{v_0}{i\pi}\int \frac{e^{-\kappa_1 a^2 t}}{a}\,da,$$

over the path (P), and we know that this is equal to v_0.

We shall prove below that the roots of the equation

$$F(a)\equiv\sigma\cos aa\sin\mu a(b-a)+\sin aa\cos\mu a(b-a)=0 \quad\text{.......(9)}$$

are infinite in number, all real, and not repeated, and it is clear that to each positive root there is an equal and opposite negative root.

Then using Fig. 15, as before, it will be seen that the values of v_1 and v_2 given by (6) and (8) satisfy the initial conditions (3) and (3′).

Finally, the solution is obtained as an infinite series.

For we have, from (6) and (8),

$$\left.\begin{aligned} v_1&=\frac{v_0}{2i\pi}\int\frac{\sin ax}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}\,da,\\ v_2&=\frac{v_0}{2i\pi}\int\frac{\sigma\cos aa\sin\mu a(x-a)+\sin aa\cos\mu a(x-a)}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}\,da,\end{aligned}\right\}$$

over the path (Q) of Fig. 17.

Hence $$v_1=v_0\left\{\frac{x}{\sigma\mu(b-a)+a}+2\sum_1^\infty\frac{\sin a_n x}{F'(a_n)}\frac{e^{-\kappa_1 a_n^2 t}}{a_n}\right\} \quad\text{..........(10)}$$

and

$$v_2=v_0\left\{\frac{\sigma\mu(x-a)+a}{\sigma\mu(b-a)+a}+2\sum_1^\infty\frac{\sigma\cos a_n a\sin\mu a_n(x-a)+\sin a_n a\cos\mu a_n(x-a)}{F'(a_n)}\frac{e^{-\kappa_1 a_n^2 t}}{a}\right\}, \quad (11)$$

the summation being taken over the positive roots of (9).

When the conditions at the surface are of the form discussed in the previous pages, the method of this section can be applied with success to the solution of the problem.

106. It remains to discuss the roots of the equation $F(a)=0$. These are the common roots, if any, of

$$\left.\begin{aligned}\sin aa&=0,\\ \sin\mu a(b-a)&=0,\end{aligned}\right\} \quad\text{..................................(1)}$$

and the roots of $$\sigma\cot aa+\cot\mu a(b-a)=0. \quad\text{..............................(2)}$$

The equations (1) will have common roots other than zero only if $\mu(b-a)/a$ is rational: and if $\mu(b-a)/a$ is small, these values of aa will be large.

From the curves $$\left.\begin{aligned}y&=\sigma\cot x,\\ y&=-\cot\mu\frac{(b-a)}{a}x,\end{aligned}\right\}$$

it is clear that there are an infinite number of real roots of (2), and their position can be determined. They are symmetrically placed with regard to the origin and they are not repeated.

Further (2) cannot have a pure imaginary root, $i\eta$ say, since

$$\sigma \coth a\eta + \coth \mu\eta(b-a)$$

cannot be zero.

We shall now show that there are no imaginary roots of (2) of the form $\xi \pm i\eta$.

Consider the function U defined as follows:

$$U = \begin{cases} U_1 = \sin \alpha x, & 0 < x < a, \\ U_2 = \dfrac{\sin \alpha a}{\sin \mu\alpha(b-a)} \sin \mu\alpha(b-x), & a < x < b, \end{cases}$$

where α is a root of $F(\alpha)=0$.

Then we have
$$\frac{d^2U_1}{dx^2} + \alpha^2 U_1 = 0, \qquad 0 < x < a, \quad \text{.....(12)}$$
$$\frac{d^2U_2}{dx^2} + \mu^2\alpha^2 U_2 = 0, \quad a < x < b. \quad \text{.....(13)}$$

Also $U_1 = 0$, when $x=0$; $U_2=0$, when $x=b$.

And $U_1 = U_2$, when $x=a$.

Further, when $x=a$,

$$K_1\frac{dU_1}{dx} - K_2\frac{dU_2}{dx}$$
$$= \frac{\alpha}{\sin \mu\alpha(b-a)}\{K_1 \cos \alpha a \sin \mu\alpha(b-a) + K_2\mu \sin \alpha a \cos \mu\alpha(b-a)\}$$
$$=0, \text{ since } K_1 = \sigma\mu K_2.$$

Let α and β be two different roots of $F(\alpha)=0$.

Let U_1, U_2 have the values given above, and let V_1, V_2 be the corresponding expressions when β is put in the place of α. Then we have from (12) and (13),

$$\frac{\mu}{\sigma}(\alpha^2-\beta^2)\int_a^b U_2V_2\,dx + \frac{1}{\sigma\mu}\int_a^b (U_2''V_2 - U_2V_2'')\,dx = 0. \quad \text{.....(14)}$$
$$(\alpha^2-\beta^2)\int_0^a U_1V_1\,dx + \int_0^a (U_1''V_1 - U_1V_1'')\,dx = 0. \quad \text{.....(15)}$$

Therefore

$$(\alpha^2-\beta^2)\left[\int^a U_1V_1\,dx + \frac{\mu}{\sigma}\int_a^b U_2V_2\,dx\right]$$
$$= \frac{1}{\sigma\mu}\int_a^b (U_2V_2'' - U_2''V_2)\,dx + \int_0^a (U_1V_1'' - U_1''V_1)\,dx$$
$$= \frac{K_2}{K_1}\Big[U_2V_2' - U_2'V_2\Big]_a^b + \Big[U_1V_1' - U_1'V_1\Big]_0^a$$
$$= \frac{1}{K_1}[U_1(K_1V_1' - K_2V_2')_a + V_1(K_2U_2' - K_1U_1')_a].$$

But we have seen that

$$\left.\begin{aligned} K_1\frac{dU_1}{dx} &= K_2\frac{dU_2}{dx}, \\ K_1\frac{dV_1}{dx} &= K_2\frac{dV_2}{dx}, \end{aligned}\right\} \text{ when } x=a.$$

It follows that

$$(\alpha^2-\beta^2)\left\{\int_0^a U_1V_1\,dx+\frac{\mu}{\sigma}\int_a^b U_2V_2\,dx\right\}=0. \quad \text{..................(16)}$$

It is clear from (16) that α, β cannot be of the form $\xi \pm i\eta$, since U_1, V_1 and U_2, V_2 would be conjugate imaginaries and

$$\int_0^a U_1V_1\,dx+\frac{\mu}{\sigma}\int_a^b U_2V_2\,dx$$

would be positive.

FLOW OF HEAT IN A SPHERE.

107. When the initial and surface conditions in a homogeneous sphere are such that the isothermal surfaces are concentric spheres, we have seen that the equations for the temperature can be reduced to those for a rod whose length is equal to the radius (§ 64). When the initial temperature is zero and the surface is kept at the constant temperature v_0, the equations for v are as follows:

$$\frac{\partial}{\partial t}(vr)=\kappa\frac{\partial^2}{\partial r^2}(vr), \quad 0<r<a,$$

$$v=0, \quad \text{when } t=0,$$

$$v=v_0, \quad \text{when } r=a.$$

Thus, from § 100 we have

$$vr=\frac{v_0 a}{i\pi}\int\frac{\sin \alpha r}{\sin \alpha a}\,\frac{e^{-\kappa\alpha^2 t}}{\alpha}\,d\alpha,$$

over the standard path (P) of Fig. 14.

Also

$$\begin{aligned}
a\left(\frac{\partial v}{\partial r}\right)_{r=a}+v_0 &= \frac{v_0 a}{i\pi}\int \cot \alpha a\, e^{-\kappa\alpha^2 t}\,d\alpha, \text{ over the path } (P),\\
&= -\frac{v_0 a}{\pi}\int\frac{1+e^{2i\alpha a}}{1-e^{2i\alpha a}}\,e^{-\kappa\alpha^2 t}\,d\alpha\\
&= -\frac{v_0 a}{\pi}\int e^{-\kappa\alpha^2 t}\,d\alpha-\frac{2v_0 a}{\pi}\sum_1^\infty\int e^{2in\alpha a}\,e^{-\kappa\alpha^2 t}\,d\alpha\\
&= \frac{2v_0 a}{\pi}\int_0^\infty e^{-\kappa\alpha^2 t}\,d\alpha+\frac{4v_0 a}{\pi}\sum_1^\infty\int_0^\infty \cos 2n\alpha a\, e^{-\kappa\alpha^2 t}\,d\alpha\\
&= \frac{v_0 a}{\sqrt{(\pi\kappa t)}}\left\{1+2\sum_1^\infty e^{-\frac{n^2a^2}{\kappa t}}\right\}.
\end{aligned}$$

Thus,

$$\left(\frac{\partial v}{\partial r}\right)_{r=a}=\frac{v_0}{\sqrt{(\pi\kappa t)}}-\frac{v_0}{a},$$

approximately when t is large.

And when the initial temperature of the sphere is v_0 and the surface temperature zero, we have the approximation

$$\left(\frac{\partial v}{\partial r}\right)_{r=a}=\frac{v_0}{a}-\frac{v_0}{\sqrt{(\pi\kappa t)}}.$$

Let t_1 and t_2 be the times required for subsidence to a certain temperature gradient at the surface in the plane problem (§ 28) and this spherical problem.

Then we have $$\frac{v_0}{\sqrt{(\pi\kappa t_1)}}=\frac{v_0}{\sqrt{(\pi\kappa t_2)}}-\frac{v_0}{a}.$$

Therefore $$\frac{1}{\sqrt{t_1}}=\frac{1}{\sqrt{t_2}}-\frac{\sqrt{(\pi\kappa)}}{a}.$$

Let $$t_2=t_1+\tau.$$

Then $$\frac{\tau}{t_1}=\frac{2}{a}\sqrt{(\pi\kappa t_1)},\text{ approximately.}$$

But the gradient of 1 degree in 50 feet adopted by Kelvin is the same as 1 degree in 2743 cm.

Also $a=6{\cdot}38\times10^8$ cm. and $v_0=4000°$ C.

It follows that $\dfrac{\tau}{t_1}=\dfrac{1}{29}$.*

108. Sphere of radius b composed of two different Materials. From r=0 to r=a of one: from r=a to r=b of another. Surface r=b kept at Constant Temperature v_0. Initial Temperature Zero.†

As in § 105, let v_1, K_1, c_1, ρ_1 refer to the part from $r=0$ to $r=a$, and v_2, K_2, c_2, ρ_2 to that from $r=a$ to $r=b$.

Also let $\kappa_1=K_1/c_1\rho_1$ and $\kappa_2=K_2/c_2\rho_2$.

Then the equations to be solved are as follows:

$$\text{(1)}\quad \frac{\partial v_1}{\partial t}=\kappa_1\left(\frac{\partial^2 v_1}{\partial r^2}+\frac{2}{r}\frac{\partial v_1}{\partial r}\right),\ 0<r<a: \quad \frac{\partial v_2}{\partial t}=\kappa_2\left(\frac{\partial^2 v_2}{\partial r^2}+\frac{2}{r}\frac{\partial v_2}{\partial r}\right),\ a<r<b. \text{(1)}$$

$$\text{(2)}\quad v_2=v_0,\text{ when } r=b.$$

$$\text{(3)}\quad v_1=0,\text{ when } t=0,\ 0<r<a: \qquad v_2=0,\text{ when } t=0,\ a<r<b. \ \ldots\text{(3)}$$

$$\text{(4)}\quad v_1=v_2,\text{ when } r=a.$$

$$\text{(5)}\quad K_1\frac{\partial v_1}{\partial r}=K_2\frac{\partial v_2}{\partial r},\text{ when } r=a.$$

* This agrees with Heaviside's result, *loc. cit.*, p. 14.

† Heaviside solved this problem by his "operational method" (cf. *loc. cit.*, p. 19), but he has not published his investigation. See also the papers by Bromwich and the author (*Cambridge, Proc. Phil. Soc.*, 20) referred to on p. 201.

On putting $v_1 r = u_1$ and $v_2 r = u_2$, these reduce to:

(6) $\dfrac{\partial u_1}{\partial t} = \kappa_1 \dfrac{\partial^2 u_1}{\partial r^2}, \quad 0 < r < a$: $\qquad \dfrac{\partial u_2}{\partial t} = \kappa_2 \dfrac{\partial^2 u_2}{\partial r^2}, \quad a < r < b$. ....(6')

(7) $u_1 = 0$, when $r = 0$: $\qquad u_2 = bv_0$, when $r = b$. ....(7')

(8) $u_1 = 0$, when $t = 0$, $0 < r < a$: $\qquad u_2 = 0$, when $t = 0$, $a < r < b$. (8')

(9) $u_1 = u_2$, when $r = a$.

(10) $K_1\left(a\dfrac{\partial u_1}{\partial r} - u_1\right) = K_2\left(a\dfrac{\partial u_2}{\partial r} - u_2\right)$, when $r = a$.

It is clear that

$$u_1 = A_1 \sin \alpha r \, e^{-\kappa_1 \alpha^2 t},$$
$$u_2 = (A_2 \sin \mu\alpha(r-a) + B_2 \sin \mu\alpha(b-r))\, e^{-\kappa_1 \alpha^2 t},$$

where $\mu = \sqrt{(\kappa_1/\kappa_2)}$, satisfy (6) and (6').

They also satisfy (9) and (10), provided that

$$\left.\begin{aligned} A_1 \sin \alpha a &= B_2 \sin \mu\alpha(b-a), \\ K_1 A_1[\alpha a \cos \alpha a - \sin \alpha a] &= K_2[\mu\alpha a(A_2 - B_2 \cos \mu\alpha(b-a) \\ &\qquad - B_2 \sin \mu\alpha(b-a)]. \end{aligned}\right\}$$

Therefore

$$\alpha a A_2 - B_2\left[\alpha a \cos \mu\alpha(b-a) + \frac{1}{\mu}\sin \mu\alpha(b-a)\right]$$
$$= \sigma(\alpha a \cos \alpha a - \sin \alpha a)\, A_1,$$

where $K_1 = K_2 \mu\sigma$.

Thus we take

$$A_2 = \frac{\sigma \cos \alpha a \sin \mu\alpha(b-a) + \sin \alpha a \cos \mu\alpha(b-a) + \dfrac{1-\mu\sigma}{\mu\alpha a}\sin \alpha a \sin \mu\alpha(b-a)}{\sin \mu\alpha(b-a)} A_1,$$

$$B_2 = \frac{\sin \alpha a}{\sin \mu\alpha(b-a)} A_1.$$

Introducing the path (P) of Fig. 14, and choosing a suitable value for A_1, we are led to the solutions:

$$u_1 = \frac{bv_0}{i\pi}\int \frac{\sin \alpha r}{F(\alpha)} \frac{e^{-\kappa_1 \alpha^2 t}}{\alpha} d\alpha, \quad \text{.......................................(11)}$$

$$u_2 = \frac{bv_0}{i\pi}\int \left\{\frac{\sin \mu\alpha(r-a)}{\sin \mu\alpha(b-a)} + \frac{\sin \alpha a}{F(\alpha)} \frac{\sin \mu\alpha(b-r)}{\sin \mu\alpha(b-a)}\right\} \frac{e^{-\kappa_1 \alpha^2 t}}{\alpha} d\alpha, \quad (12)$$

where $F(\alpha) = \sigma \cos \alpha a \sin \mu\alpha(b-a) + \sin \alpha a \cos \mu\alpha(b-a)$

$$+ \frac{1-\mu\sigma}{\mu\alpha a}\sin \alpha a \sin \mu\alpha(b-a),$$

the integrals being taken over the path (P).

The value of u_2 given in (12) reduces to

$$u_2=\frac{bv_0}{i\pi}\int\frac{\sigma\cos aa\sin\mu a(r-a)+\sin aa\cos\mu a(r-a)+\dfrac{1-\mu\sigma}{\mu aa}\sin aa\sin\mu a(r-a)}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}\,da, \tag{13}$$

over the path (P).

The values of u_1 and u_2 given in (11) and (13) satisfy all the conditions of our problem. For, from the way in which they have been built up, they obviously satisfy (6) and (6′), and the conditions (7), (9) and (10), which hold when $r=0$ and $r=a$.

Further, putting $r=b$ in (13), we have

$$\frac{bv_0}{i\pi}\int\frac{e^{-\kappa_1 a^2 t}}{a}\,da,$$

over the path (P), and we know this is equal to bv_0.

It can be proved just as on p. 215 that the roots of the equation

$$F(a)\equiv\sigma\cos aa\sin\mu a(b-a)+\sin aa\cos\mu a(b-a)+\frac{1-\mu\sigma}{\mu aa}\sin aa\sin\mu a(b-a)=0 \tag{14}$$

are infinite in number, all real and not repeated, and to each positive root there is an equal and opposite negative root.

Assuming this to be the case, it follows from Fig. 16, as before, that the values of u_1 and u_2 in (11) and (13) satisfy the initial conditions (8) and (8′).

Finally the solution is obtained as an infinite series.

For we have, from (11) and (13),

$$\left.\begin{aligned}u_1&=\frac{bv_0}{2i\pi}\int\frac{\sin ar}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}\,da,\\ u_2&=\frac{bv_0}{2i\pi}\int\frac{\sigma\cos aa\sin\mu a(r-a)+\sin aa\cos\mu a(r-a)+\dfrac{1-\mu\sigma}{\mu aa}\sin aa\sin\mu a(r-a)}{F(a)}\frac{e^{-\kappa_1 a^2 t}}{a}\,da,\end{aligned}\right\}$$

the integrals now being taken over the path (Q) of Fig. 17.

Hence

$$\left.\begin{aligned}u_1&=rv_0+2bv_0\sum_1^\infty\frac{\sin a_n r}{F'(a_n)}\frac{e^{-\kappa_1 a_n^2 t}}{a_n},\\ u_2&=rv_0+2bv_0\\ &\times\sum_1^\infty\frac{\sigma\cos a_n a\sin\mu a_n(r-a)+\sin a_n a\cos\mu a_n(r-a)+\dfrac{1-\mu\sigma}{\mu a_n a}\sin a_n a\sin\mu a_n(r-a)}{F'(a_n)}\frac{e^{-\kappa_1 a_n^2 t}}{a_n}\end{aligned}\right\} \tag{15}$$

the summation being taken over the positive roots of the equation (14).

When the surface is kept at a temperature Ct, Ct^2, etc., or $C\cos\omega t$, etc., or radiation takes place at the surface into a medium at a constant temperature, or at one of these just named, the method of this section can be applied with success to the solution of the problem.

109. When the sphere of § 108 has its surface kept at zero temperature and the initial temperature of the whole solid is v_0, the temperatures in the inner and outer parts follow from § 108 (15) and are given respectively by

$$v_1 = -\frac{2bv_0}{r}\sum_1^\infty \frac{\sin \alpha_n r}{F'(\alpha_n)}\frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n},$$

$$v_2 = -\frac{2bv_0}{r}\sum_1^\infty \frac{\sigma\cos\alpha_n a\sin\mu\alpha_n(r-a)+\sin\alpha_n a\cos\mu\alpha_n(r-a) + \dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin\alpha_n a\sin\mu\alpha_n(r-a)}{F'(\alpha_n)}\frac{e^{-\kappa_1\alpha_n^2 t}}{\alpha_n}.$$

The summation is taken over the positive roots of the equation $F(\alpha)=0$ [§ 108 (14)].

Thus $\left(\dfrac{\partial v_2}{\partial r}\right)_{r=b}$

$$= 2\mu v_0\sum_1^\infty \frac{\sigma\cos\alpha_n a\cos\mu\alpha_n(b-a)+\dfrac{1-\mu\sigma}{\mu\alpha_n a}\sin\alpha_n a\cos\mu\alpha_n(b-a) - \sin\alpha_n a\sin\mu\alpha_n(b-a)}{F'(\alpha_n)}e^{-\kappa_1\alpha_n^2 t}.$$

The equation $F(\alpha)=0$ will be satisfied by the common roots, if any, of

$$\left.\begin{aligned}\sin\alpha a&=0,\\ \sin\mu\alpha(b-a)&=0,\end{aligned}\right\} \quad\text{......................................(1)}$$

and by the roots of

$$\sigma\cot\alpha a+\cot\mu\alpha(b-a)+\frac{1-\mu\sigma}{\mu\alpha a}=0. \quad\text{..........................(2)}$$

The equations (1) will have common roots other than zero only if $\mu(b-a)/a$ is rational: and if $\mu(b-a)/a$ is small, these values of αa will be large.

With the data adopted by Perry and Heaviside in c.g.s. units, and the above notation:

$$a=6{\cdot}38\times10^8,\qquad b-a=4\times10^6,\qquad v_0=4\times10^3,$$
$$K_1=\cdot47,\qquad K_2=\cdot00595,$$
$$c_1\rho_1=2{\cdot}86,\qquad c_2\rho_2=\cdot507,$$
$$\kappa_1=\frac{K_1}{c_1\rho_1}=\cdot1643,\qquad \kappa_2=\frac{K_2}{c_2\rho_2}=\cdot0117,$$
$$\mu=\sqrt{(K_1/K_2)}=3{\cdot}742,\qquad \sigma=\sqrt{((K_1c_1\rho_1)/(K_2c_2\rho_2))}=21{\cdot}1,\qquad \mu\sigma=79.$$

Also the equation (2) above becomes

$$21{\cdot}1\cot x+\cot(2{\cdot}35\times10^{-3}x)-\frac{20{\cdot}8}{x}=0,\ \text{where}\ \alpha a=x.$$

The first root is $x_1=2{\cdot}9871$; the second root is $x_2=5{\cdot}980$; and the later roots approach 3π, 4π, etc. Also t is large.

Thus we may take the first term as a good enough approximation for the gradient $\left(\frac{\partial v_2}{\partial r}\right)$, when $r=b$.

In the Age of the Earth problem (cf. § 28), the time of cooling to the gradient 1 degree in 50 ft. or 1 degree in 2743 cm. is required.

It will be found* that the equation

$$\frac{1}{2743}=2\mu v_0 \frac{\sigma\cot\alpha_1 a\cot\mu\alpha_1(b-a)+\dfrac{1-\mu\sigma}{\mu\alpha_1 a}\cot\mu\alpha_1(b-a)-1}{\sigma a\operatorname{cosec}^2\alpha_1 a+\mu(b-a)\operatorname{cosec}^2\mu\alpha_1(b-a)+\dfrac{1-\mu\sigma}{\mu\alpha_1^2 a}}\, e^{-\kappa_1\alpha_1^2 t}$$

gives for the time *in years* $9{\cdot}02\times10^9$. This is about one-fifth of the result found in § 99 for the corresponding plane problem.

FLOW OF HEAT IN A CIRCULAR CYLINDER.

110. The method employed in the preceding sections is also applicable to the case of the circular cylinder, when the temperature v depends only upon r and t.

In this case the equation of conduction becomes

$$\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right).$$

We shall refer only very briefly to certain problems for the Cylinder, as with a little practice these can be easily solved.

I. *Circular Cylinder of radius a. Surface at Constant Temperature v_0. Initial Temperature Zero.*

It is clear that

$$v=\frac{v_0}{i\pi}\int\frac{J_0(ar)}{J_0(aa)}\frac{e^{-\kappa a^2 t}}{a}\,da,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

over the standard path (P) of Fig. 14, satisfies all the conditions of the problem, for the zeros of $J_0(z)$ are known to be real and not repeated,† and when $t=0$ we can use Fig. 16 as before.

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* Cf. *Proc. Camb. Phil. Soc.*, **20**, p. 404, 1921.

† Cf. Watson, *loc. cit.*, §§ 15. 21, 15. 25.

The solution given in (1) can be put in the form

$$v=\frac{v_0}{2i\pi}\int\frac{J_0(ar)}{J_0(aa)}\frac{e^{-\kappa a^2t}}{a}\,da,$$

over the path (Q) of Fig. 17.

And, finally, we obtain

$$v=v_0\left[1+\frac{2}{a}\sum_1^\infty\frac{J_0(a_nr)}{J_0'(a_na)}\frac{e^{-\kappa a_n^2t}}{a_n}\right],$$

the summation being taken over the positive roots of $J_0(aa)=0$. (Cf. § 57, I.)

II. *The Same Solid. Surface Temperature Ct.*

Here our solution is

$$v=-\frac{C}{i\kappa\pi}\int\frac{J_0(ar)}{J_0(aa)}\frac{e^{-\kappa a^2t}}{a^3}\,da, \quad\text{.....................(1)}$$

over the path (P) of Fig. 14.

This reduces to

$$v=-\frac{C}{\kappa}\left[\frac{1}{4}(a^2-r^2-4\kappa t)+\frac{2}{a}\sum_1^\infty\frac{J_0(a_nr)}{J_0'(a_na)}\frac{e^{-\kappa a_n^2t}}{a_n^3}\right], \quad\text{.........(2)}$$

the summation being as before.

For certain applications* the mean temperature over the cylinder is required, *i.e.*

$$\frac{1}{\pi a^2}\int_0^a 2\pi rv\,dr, \quad\text{or}\quad \frac{2}{a^2}\int_0^a vr\,dr.$$

This could be obtained from (2), or more directly, at once from (1).

By the latter method, we see that

$$\text{the mean temperature}=\frac{2}{a^2}\int_0^a vr\,dr$$

$$=-\frac{2C}{i\kappa\pi a^2}\int\frac{e^{-\kappa a^2t}}{a^3J_0(aa)}\left(\int_0^a rJ_0(ar)\,dr\right)da,$$

integrating over the path (P).

But $$\int_0^a rJ_0(ar)\,dr=-\frac{a}{a}J_0'(aa).$$

$$\text{Thus the mean temperature}=\frac{2C}{i\kappa\pi a}\int\frac{e^{-\kappa a^2t}}{a^4}\frac{J_0'(aa)}{J_0(aa)}\,da, \text{ over the path } (P),$$

$$=\frac{C}{i\kappa\pi a}\int\frac{e^{-\kappa a^2t}}{a^4}\frac{J_0'(aa)}{J_0(aa)}\,da, \text{ over the path } (Q),$$

$$=C\left[t-\frac{a^2}{8\kappa}+\frac{4}{a^2\kappa}\sum_1^\infty\frac{e^{-\kappa a_n^2t}}{a_n^4}\right].$$

---

* Cf. Bromwich, *Phil. Mag., London* (Ser. 6), 37, p. 413, 1919. The mean temperature over a sphere follows from § 102 in the same way.

III. *The Same Solid. Surface Temperature $C\cos\omega t$.*

Here our solution is

$$v=\frac{C}{i\pi}\int\frac{J_0(\alpha r)}{J_0(\alpha a)}e^{-\kappa\alpha^2 t}\frac{\alpha^3 d\alpha}{\alpha^4+\omega^2/\kappa^2},$$

over the path (P) of Fig. 14,* and this can be reduced, as before, to an infinite series.

IV. *The Same Solid. Radiation at the Surface into a Medium at Constant Temperature v_0.*

Here our solution is

$$v=\frac{hv_0}{i\pi}\int\frac{J_0(\alpha r)}{\frac{d}{da}J_0(\alpha a)+hJ_0(\alpha a)}\frac{e^{-\kappa\alpha^2 t}}{\alpha}d\alpha,$$

over the path (P) of Fig. 14, which can be reduced to an infinite series, as before.

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* The footnote on p. 203 applies here also.

CHAPTER XII

INTEGRAL EQUATIONS AND THE EQUATION OF CONDUCTION

111. Introductory.

An integral equation* is one which involves an unknown function under the sign of integration. The equation

$$\phi(x)=\lambda\int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \quad\text{.........................(1)}$$

is called a *homogeneous* integral equation. The function $\phi(x)$ is the unknown function. The known function $K(x, \xi)$ is called the *kernel* (or nucleus) of the equation. It will usually be a continuous function of (x, ξ) in the region $a \leqq x \leqq b$, $a \leqq \xi \leqq b$ with which we are concerned. More general conditions for $K(x, \xi)$ are referred to in the works cited in the preceding footnote.

In the theory of integral equations it is shown that the only continuous function which satisfies (1) is $\phi(x)=0$, when λ is not a zero of a certain function $D(\lambda)$ associated with this equation. The roots of the equation $D(\lambda)=0$ are called the *characteristic*

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* For a discussion of the elementary theory of integral equations, reference may be made to Bôcher's *Introduction to the Study of Integral Equations* (Cambridge Tracts in Mathematics, No. 10). There is a short chapter on the subject in Whittaker and Watson's *Modern Analysis*, and a much fuller treatment in Goursat's *Cours d'Analyse*, T. III.

For the applications, the following works will be found specially useful:

Horn, *Einführung in die Theorie der partiellen Differentialgleichungen*, Leipzig, 1910. (Sammlung Schubert, LX.)

Kneser, *Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik*, Braunschweig, 1911.

Heywood et Fréchet, *L'Équation de Fredholm et ses Applications à la Physique Mathématique*, Paris, 1912.

Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig, 1912.

Vivanti, *Elementi della teoria delle equazioni integrali lineari*, Milano, 1916.

numbers of this kernel. When $K(x, \xi)$ is a symmetric function of x, ξ, the characteristic numbers are known to be all real.

Let $\phi(x)$ be a continuous function, not identically zero, satisfying (1) for a value of λ which is a root of $D(\lambda)=0$. It is said to be a *characteristic function* corresponding to this characteristic number λ. Also it is said to be *normalised* if $\int_a^b \phi^2 dx=1$.

It is known that if $\phi_m(x)$ corresponds to λ_m and $\phi_n(x)$ to another characteristic number λ_n for a symmetric kernel, then $\int_a^b \phi_m \phi_n dx=0$. Functions for which $\int_a^b \phi(x)\psi(x)\,dx=0$ are said to be *orthogonal*, and they are normalised orthogoual functions if, in addition, $\int_a^b \phi^2 dx=\int_a^b \psi^2(x)\,dx=1$.

In the case of finite symmetric kernels there is an upper limit to the number of orthogonal characteristic functions which correspond to the same value of λ and every other characteristic function, for that value of λ, is linearly dependent on these orthogonal functions. There will thus be a set of characteristic functions

$$\phi_1(x), \quad \phi_2(x), \ldots$$

finite or infinite in number as the case may be, orthogonal to each other and normalised, such that every characteristic function of this symmetric kernel is linearly dependent upon a finite number of them. Such a system is spoken of as a *complete* orthogonal system of normalised characteristic functions of this kernel.

With regard to such a system we have the following theorem:

Let $\phi_1(x)$, $\phi_2(x)$, ... be a complete orthogonal system of normalised characteristic functions for the homogeneous integral equation with finite symmetric kernel

$$\phi(x)=\lambda\int_a^b K(x, \xi)\,\phi(\xi)\,d\xi,$$

and λ_1, λ_2, ... the corresponding characteristic numbers. If the series

$$\sum_1^\infty \frac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n}$$

is uniformly convergent in the region $a \leqq x \leqq b$, $a \leqq \xi \leqq b$, then its sum is $K(x, \xi)$ at every point at which the kernel is continuous.

Of this theorem we shall have to make frequent use in the applications of integral equations of this type to the solution of problems in the conduction of heat.

112. Integral Equations and Linear Flow of Heat.

Consider now the temperature problem for the rod of length l, the ends kept at zero, the initial temperature being the arbitrary function $f(x)$.

Then we have the equations:

$$\left.\begin{aligned} &\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}, \quad 0<x<l,\\ &v=0, \text{ when } x=0 \text{ and } x=l,\\ &v=f(x), \text{ when } t=0.\end{aligned}\right\} \quad \text{......................(1)}$$

Putting $v=e^{-\kappa\lambda t}\phi(x)$, we have

$$\left.\begin{aligned} &\frac{d^2\phi}{dx^2}+\lambda\phi=0,\\ &\phi=0, \text{ when } x=0 \text{ and } x=l.\end{aligned}\right\} \quad \text{......................(2)}$$

Thus $\lambda=n^2\pi^2/l^2$, and choosing ϕ so that $\int_0^l \phi^2(x)\,dx=1$, the normalised function ϕ_n is $\sqrt{\left(\frac{2}{l}\right)}\sin\frac{n\pi}{l}x$, n being a positive integer.

Also the functions ϕ are orthogonal, since $\int_0^l \phi_m\phi_n\,dx=1$, $m\neq n$.
= 0

Now there is a continuous function $K(x, \xi)$, which satisfies the equation for steady temperature ($v''=0$), and the same boundary conditions, while its differential coefficient with regard to x, denoted by $K'(x, \xi)$, is continuous except at $x=\xi$, where it is discontinuous in such a way that $\Big[K'(x, \xi)\Big]_{\xi+0}^{\xi-0}=1$. This function is called the Green's function for the steady temperature equation.* It is the steady temperature due to a constant source of a certain strength at the point $x=\xi$, and in the case given above it is clear that all the conditions are satisfied by

$$\left.\begin{aligned} K(x, \xi)&=x(1-\xi/l), \quad 0<x<\xi,\\ &=\xi(1-x/l), \quad \xi<x<l.\end{aligned}\right\} \quad \text{..............(3)}$$

Let $\qquad F(x)=\phi(x)K'(x, \xi)-\phi'(x)K(x, \xi)$,

where $\phi(x)$ is given by (2).

* The Green's functions employed in the application of Integral Equations to the Conduction of Heat must not be confused with those to which the same term was applied in Chapter X.

Then $F(x)$ is discontinuous when $x=\xi$, but it is otherwise continuous in $(0, l)$.

Also $$F'(x)=\lambda K(x, \xi)\phi(x), \text{ except when } x=\xi.$$

Therefore $$\int_0^l F'(x)\,dx=\lambda\int_0^l K(x, \xi)\phi(x)\,dx.$$

But $$\int_0^l F'(x)=\int_0^{\xi} F'(x)\,dx+\int_{\xi}^l F'(x)\,dx$$
$$=\Big[F(x)\Big]_{\xi+0}^{\xi-0}, \quad \text{since } F(0)=F(l)=0.$$

It follows that
$$\phi(\xi)\Big[K'(x, \xi)\Big]_{\xi+0}^{\xi-0}=\lambda\int_0^l K(x, \xi)\phi(x)\,dx.$$

Thus $$\phi(\xi)=\lambda\int_0^l K(x, \xi)\phi(x)\,dx,$$

and since $K(x, \xi)$ is a symmetric function of x, ξ, we have

$$\phi(x)=\lambda\int_0^l K(x, \xi)\phi(\xi)\,d\xi \quad\text{.............................(4)}$$

Thus *the functions* $\phi(x)$ *of* (2) *occur as characteristic functions of the homogeneous integral equation* (4).

The converse is also true. Every continuous solution $\phi(x)$ of the integral equation (4) satisfies the equations (2).

We start with $$\phi(x)=\lambda\int_0^l K(x, \xi)\phi(\xi)\,d\xi,$$

where $K(x, \xi)$ is given by (3).

Since $K'(x, \xi)$ is discontinuous when $x=\xi$, we cannot differentiate under the sign of integration, if we rely only upon the theorem proved in *F.S.* § 78. But it is easy to extend that theorem to such a case as this by the following method:*

Let $K(x, \xi)$ be given by (3) above, and let

$$f(x, \xi)=K(x, \xi)=x(1-\xi/l), \quad \text{when } x<\xi$$
$$=K(x, \xi)+(x-\xi), \quad \text{when } x>\xi.$$

Then $f(x, \xi)=x(1-\xi/l)$, and this function is continuous in (x, ξ)

* For a discussion of the question of differentiation under the sign of integration, see Hardy, *Q. J. Math., London*, 32, p. 66, 1901, and *Mess. Math., Cambridge*, 31, 1902; 33, 1904.

in the region $0 \leqq x \leqq l$, $0 \leqq \xi \leqq l$. Also $\dfrac{\partial f}{\partial x}$ is continuous in this region.

Further,

$$\phi(x) = \lambda \int_0^x [f(x, \xi) - (x - \xi)]\, \phi(\xi)\, d\xi + \lambda \int_x^l f(x, \xi)\, \phi(\xi)\, d\xi$$

$$= \lambda \int_0^l f(x, \xi)\, \phi(\xi)\, d\xi - \lambda \int_0^x (x - \xi)\, \phi(\xi)\, d\xi.$$

Therefore

$$\phi'(x) = \lambda \int_0^l \frac{\partial f}{\partial x} \phi(\xi)\, d\xi - \lambda \int_0^x \phi(\xi)\, d\xi$$

$$= \lambda \left[\int_0^x \left(\frac{\partial K}{\partial x} + 1 \right) \phi(\xi)\, d\xi + \int_x^l \frac{\partial K}{\partial x} \phi(\xi)\, d\xi \right] - \lambda \int_0^x \phi(\xi)\, d\xi$$

$$= \lambda \int_0^l \frac{\partial K}{\partial x} \phi(\xi)\, d\xi. \qquad (5)$$

Again, we cannot differentiate (5) under the sign of integration if we rely only upon the theorem of *F.S.*, § 78, since the integrand is discontinuous when $x = \xi$. But we can extend that theorem to such a case as this in a similar way:

$$\text{Let} \quad \left.\begin{aligned} g(x, \xi) &= K'(x, \xi) = 1 - \xi/l, && \text{when } x < \xi, \\ &= K'(x, \xi) + 1, && \text{when } x > \xi. \end{aligned}\right\}$$

Then $g(x, \xi) = 1 - \xi/l$ in the given region,

and $$\phi'(x) = \lambda \int_0^l g(x, \xi)\, \phi(\xi)\, d\xi - \lambda \int_0^x \phi(\xi)\, d\xi.$$

Therefore $$\phi''(x) = \lambda \int_0^l \frac{\partial g}{\partial x} \phi(\xi)\, d\xi - \lambda\, \phi(x)$$

$$= -\lambda\, \phi(x), \quad \text{since } \frac{\partial g}{\partial x} = 0.$$

Finally, since $K(x, \xi)$ vanishes when $x = 0$ and $x = l$, the functions ϕ defined by (4) also vanish when $x = 0$ and $x = l$.

Thus we have established that *all the characteristic functions of the homogeneous integral equation* (4) *are solutions of the equations* (2).

In this question the characteristic numbers are $n^2\pi^2/l^2$, where n is any positive number, and a complete normalised orthogonal system of characteristic functions is $\phi_1(x)$, $\phi_2(x)$, ..., where

$$\phi_n(x) = \sqrt{\left(\frac{2}{l}\right)} \sin \frac{n\pi}{l} x.$$

But the series $$\frac{2l}{\pi^2}\sum_1^\infty \frac{1}{n^2}\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}\xi$$

is uniformly convergent in the region $0 \leqq x \leqq l$, $0 \leqq \xi \leqq l$.

It follows from the theorem enunciated at the end of § 107 that for $K(x, \xi)$ as given in (3) we have

$$K(x, \xi)=\sum_1^\infty \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n}. \quad \text{.............................(6)}$$

113. The real difficulty in the solution of the problem of linear flow of § 112 consists in establishing the possibility of the expansion of the arbitrary function in the appropriate series. From the result in (6) above, it can be shown that the arbitrary function $f(x)$ is equal to the sum of an infinite series of these characteristic functions, provided that

$$f(x)=\int_0^l K(x, \xi)\psi(\xi)\,d\xi$$

and $\psi(x)$ is bounded and integrable in $(0, l)$.*

For we are given that

$$K(x, \xi)=\sum_1^\infty \frac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n},$$

and this series is uniformly convergent in the region $0 \leqq x \leqq l$, $0 \leqq \xi \leqq l$.

It follows that $$f(x)=\int_0^l \sum_1^\infty \frac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n}\psi(\xi)\,d\xi$$

$$=\sum_1^\infty \frac{\phi_n(x)}{\lambda_n}\int_0^l \phi_n(\xi)\,\psi(\xi)\,d\xi. \qquad (F.S., \S 70.)$$

Also $$\int_0^l f(\eta)\,\phi_n(\eta)\,d\eta=\int_0^l \phi_n(\eta)\left[\int_0^l K(\eta, \xi)\,\psi(\xi)\,d\xi\right]d\eta$$

$$=\int_0^l \psi(\xi)\left[\int_0^l K(\eta, \xi)\,\phi_n(\eta)\,d\eta\right]d\xi$$

(*F.S.*, § 79)

$$=\int_0^l \psi(\xi)\left[\int_0^l K(\xi, \eta)\,\phi_n(\eta)\,d\eta\right]d\xi$$

($K(\xi, \eta)$ being symmetric)

$$=\frac{1}{\lambda_n}\int_0^l \phi_n(\xi)\,\psi(\xi)\,d\xi.$$

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* If $f(x)$ satisfies this condition, it follows from *F.S.* § 77, II. that it is continuous.

Thus when $f(x)$ can be put in the form

$$\int_0^l K(x, \xi)\, \psi(\xi)\, d\xi,$$

and $\psi(x)$ is bounded and integrable in $(0, l)$, we have

$$f(x) = \sum_1^\infty a_n \phi_n(x), \quad \text{where} \quad a_n = \int_0^l f(x')\, \phi_n(x')\, dx'.$$

Also the solution of the temperature problem of § 112 is

$$v = \sum_1^\infty a_n \phi_n(x) e^{-\kappa\lambda_n t}.$$

114. The other problems for the rod of length l can be treated in the same way.

I. *Suppose the end $x=0$ kept at zero and that radiation takes place at $x=l$ into a medium at zero.*

Then we have the equations:

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= \kappa \frac{\partial^2 v}{\partial x^2}, \\ v &= 0, \quad \text{when } x=0, \\ \frac{\partial v}{\partial x} + hv &= 0, \quad \text{when } x=l, \\ v &= f(x), \text{ when } t=0. \end{aligned}\right\}$$

Putting $v = e^{-\kappa\lambda t}\phi(x)$, we have

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

with the same conditions when $x=0$ and $x=l$.

The Green's function $K(x, \xi)$ is as follows:

$$\left.\begin{aligned} K(x, \xi) &= \left(1 - \frac{h\xi}{1+hl}\right) x, \quad 0 < x < \xi, \\ &= \left(1 - \frac{hx}{1+hl}\right) \xi, \quad \xi < x < l. \end{aligned}\right\}$$

Also the characteristic numbers are $\lambda = a^2$, where a is any positive root of $h \tan al + a = 0$.

The normalised characteristic functions are

$$\phi(x) = \frac{\sqrt{(2(a^2+h^2))}}{\sqrt{(l(a^2+h^2)+h)}} \sin ax. \quad \text{(Cf. § 65 and Ex. 2, p. 182.)}$$

The series $\sum_1^\infty \dfrac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n}$ will be found to be uniformly con-

vergent in the region $0 \leq x \leq l$, $0 \leq \xi \leq l$, and we have

$$K(x, \xi) = \sum_1^\infty \frac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n}.$$

II. *Suppose radiation takes place at both ends into a medium at zero.*

Then the conditions at the ends are

$$-\frac{\partial v}{\partial x} + hv = 0, \text{ when } x = 0,$$

$$\frac{\partial v}{\partial x} + hv = 0, \text{ when } x = l.$$

In this case

$$\left.\begin{aligned} K(x, \xi) &= \frac{(1 + h(l - \xi))(1 + hx)}{2h + lh^2}, \quad 0 < x < \xi, \\ &= \frac{(1 + h\xi)(1 + h(l - x))}{2h + lh^2}, \quad \xi < x < l. \end{aligned}\right\}$$

Also $\lambda = a^2$, where a is any positive root of $(a^2 - h^2)\tan al = 2ah$.

The normalised characteristic functions are

$$\phi(x) = \sqrt{\left(\frac{2}{(a^2 + h^2)\,l + 2h}\right)}(a \cos ax + h \sin ax). \quad \text{(Cf. § 36.)}$$

Also $$K(x, \xi) = \sum_1^\infty \frac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n}.$$

In these two cases the possibility of the expansion of the arbitrary function in the required series, when it is subject to the condition stated at the beginning of § 113, follows in the same way. The solution of the temperature problem can then be written down.

When $h = 0$ this discussion fails. For a treatment of the problem when no heat escapes at the ends, reference may be made to Kneser's work cited above, Ch. I., p. 19.

115. Fourier's Ring.

For the ring of unit radius, treated in § 12, when there is radiation at the surface into a medium at zero, we have the equations:

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} - b^2 v, \quad -\pi < x < \pi, \\ &v_{x=-\pi} = v_{x=\pi}, \\ &\left(\frac{\partial v}{\partial x}\right)_{x=-\pi} = \left(\frac{\partial v}{\partial x}\right)_{x=\pi}, \\ &v = f(x), \text{ when } t = 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1)$$

where b is a given number.

Putting $v=e^{-(\mu+b^2)t}\phi(x)$, we have

$$\frac{d^2\phi}{dx^2}+\mu\phi=0, \qquad \text{.............................(2)}$$

with the same conditions when $x=-\pi$ and $x=\pi$.

Thus we have $\mu=0$ and $\mu=n^2$, and the corresponding normalised functions are $\dfrac{1}{\sqrt{(2\pi)}}$ and $\dfrac{1}{\sqrt{\pi}}\cos nx$, $\dfrac{1}{\sqrt{\pi}}\sin nx$.

It will be noticed that to the number n^2 there correspond two orthogonal functions $\dfrac{1}{\sqrt{\pi}}\cos nx$ and $\dfrac{1}{\sqrt{\pi}}\sin nx$.

Now there is a continuous function $K(x, \xi)$, which satisfies the equation for steady temperature ($\kappa v''-b^2v=0$), and the same conditions at $x=-\pi$ and π, while its differential coefficient $K'(x, \xi)$ is continuous except at $x=\xi$, where it is discontinuous in such a way that $\Big[K'(x, \xi)\Big]_{\xi+0}^{\xi-0}=1$.

This function—the Green's Function for the equation $v''-c^2v=0$, where $c^2=b^2/\kappa$—is as follows:

$$\left.\begin{aligned} K(x, \xi)&=\frac{\cosh c(x-\xi+\pi)}{2\sinh c\pi}, \quad -\pi<x<\xi, \\ &=\frac{\cosh c(\xi-x+\pi)}{2\sinh c\pi}, \quad \xi<x<\pi. \end{aligned}\right\} \qquad \text{.........(3)}$$

Also it follows as in § 112 that, when $\phi(x)$ is given by the equations (2),

$$\phi(x)=\lambda\int_{-\pi}^{\pi}K(x, \xi)\,\phi(\xi)\,d\xi, \qquad \text{.....................(4)}$$

where $\lambda=\mu+c^2$.

And all the characteristic functions of (4) are solutions of the equations (2).

The special feature of this question is that, to each characteristic number other than $\lambda=c^2$, there correspond two orthogonal characteristic functions:

$$e.g.\ \lambda_n=c^2+n^2 \text{ has } \frac{1}{\sqrt{\pi}}\cos nx \text{ and } \frac{1}{\sqrt{\pi}}\sin nx.$$

Thus the theorem at the end of § 110 in this case leads to

$$\begin{aligned} K(x, \xi)&=\frac{1}{2\pi c^2}+\frac{1}{\pi}\sum_1^\infty\frac{\cos nx\cos n\xi+\sin nx\sin n\xi}{c^2+n^2} \\ &=\frac{1}{2\pi c^2}+\frac{1}{\pi}\sum_1^\infty\frac{\cos n(x-\xi)}{c^2+n^2}, \end{aligned}$$

provided that this series is uniformly convergent in the region $-\pi \leqq x \leqq \pi$, $-\pi \leqq \xi \leqq \pi$; a condition which is clearly satisfied.

The remarks in the previous section about the possibility of expanding an arbitrary function in the appropriate series apply also to this case. Thus we obtain the Fourier's Series for $f(x)$, under the condition stated above, for the interval $(-\pi, \pi)$ and the solution of the problem of Fourier's ring.

116. Two-Dimensional Problems.

The solution of the general problems of conduction in two and three dimensions can be made to depend upon integral equations by the introduction of a Green's function similar to that which is used in the theory of potential; but the rigorous treatment of this work is harder than that in the previous sections, since the kernel of the integral equation has an infinity in the region in which integration takes place, the integrals being surface or volume integrals, and the series are double or triple. Space does not permit of more than a slight sketch of the method; and we take first the two-dimensional case where the boundary is kept at zero.

Here we have to solve the equations:

$$\left.\begin{aligned} &\frac{\partial v}{\partial t}=\kappa\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), && \text{inside a curve } C,\\ &v=0, && \text{on the curve } C,\\ &v=f(x), \text{ when } t=0, && \text{inside the curve } C.\end{aligned}\right\} \qquad (1)$$

As before, we put $v=e^{-\kappa\lambda t}\phi(x, y)$, and we have

$$\left.\begin{aligned} \frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\lambda\phi=0, \text{ inside } C,\\ \phi=0, \text{ on } C.\end{aligned}\right\} \qquad (2)$$

The Green's function—which we shall denote by $K(x, y;\ x', y')$—is the solution of the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, which vanishes on the curve C and is finite and continuous, as also its first and second differential coefficients, inside C, except at the point (x', y'), where it becomes infinite as $-(\log r)/2\pi$ when $r\to 0$. This function is the steady temperature at the point (x, y) due to a constant line source of a certain strength at the point (x', y'), the boundary curve C being kept at zero.

We now apply Green's Theorem in two dimensions to the region

bounded by the curve C and a small circle Γ' whose centre is at (x', y'), the functions employed being $K(x, y; x', y')$ and a function $\phi(x, y)$ satisfying the equations (2).

Then we have

$$\iint (\phi\nabla^2 K - K\nabla^2\phi)\,dx\,dy = \Sigma\int\left(\phi\frac{\partial K}{\partial n} - K\frac{\partial \phi}{\partial n}\right)ds, \qquad (3)$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outward from the region of integration, and the integrals on the right are taken round the curve C and the circle Γ'.

But $\nabla^2 K = 0$ and $\nabla^2\phi + \lambda\phi = 0$ in the region through which integration takes place; and ϕ, K both vanish on the curve C; while $K(x, y; x', y')$ is infinite as $-(\log r)/2\pi$, when $r \to 0$, at the point (x', y').

Thus from (3), on letting the radius of the circle Γ' tend to zero, we have

$$\phi(x', y') = \lambda\iint K(x, y; x', y')\,\phi(x, y)\,dx\,dy, \qquad (4)$$

the double integral now being taken through the region bounded by C.

A similar application of Green's Theorem in two dimensions to the region bounded by the curve C and small circles Γ', Γ'' with centres at (x', y') and (x'', y''), the functions employed being $K(x, y; x', y')$ and $K(x, y; x'', y'')$, shows that

$$K(x', y'; x'', y'') = K(x'', y''; x', y').$$

Thus the Green's function $K(x, y; x', y')$ is a symmetric function of the two pairs (x, y) and (x', y').

It follows from (4) that

$$\phi(x, y) = \lambda\iint K(x, y; x', y')\,\phi(x', y')\,dx'\,dy', \qquad (5)$$

the double integral being taken through the region bounded by C, and (x, y) being any point in this region.

Thus the numbers λ and the functions ϕ of equations (2) enter as the characteristic numbers and characteristic functions of the homogeneous integral equation (5) with a symmetric kernel.

Also the converse is true. The theory of potential shows that when $\phi(x, y)$ is defined by (5), we have

$$\left.\begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \lambda\phi &= 0 \text{ inside } C, \\ \text{and} \qquad \phi &= 0 \text{ on } C. \end{aligned}\right\}$$

The characteristic functions corresponding to different characteristic numbers are orthogonal, and they are to be normalised by arranging that

$$\iint \phi^2(x, y)\,dx\,dy = 1,$$

the integral being taken through the given region.

The question of the expansion of the arbitrary function defining the initial temperature in a complete series of orthogonal characteristic functions, and the corresponding expansion of the Green's function, offer greater difficulty than in the case of one-dimensional problems. For a full discussion of these topics reference must be made to the works dealing with integral equations.

The argument of this section applies equally well to the case when radiation takes place into a medium at zero, the condition at the boundary being replaced by $\dfrac{\partial v}{\partial n} + hv = 0$, both in the statement of the problem and the definition of the Green's function.

117. (i) Consider the temperature problem for the rectangle

$$\left.\begin{matrix} x=0, & x=b, \\ y=0, & y=c, \end{matrix}\right\}$$

the sides kept at zero.

Here we have
$$\phi = \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y,$$
where m, n are positive integers, and

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{c^2}\right).$$

The normalised function ϕ_{mn} will be given by

$$\phi_{mn} = \frac{2}{\sqrt{(bc)}} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y.$$

Also $$K(x, y;\, x', y') = \frac{4}{\pi^2 bc}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y \sin\frac{m\pi}{b}x' \sin\frac{n\pi}{c}y'}{m^2/b^2 + n^2/c^2}\right),$$

assuming that the general theorem of § 107 applies to this case.

(ii) For the cylinder $r=a$, the surface kept at zero, we have:

$$\phi_{m,n}(r, \theta) = \frac{J_n(a_{m,n}r)\begin{matrix}\cos\\ \sin\end{matrix} n\theta}{\sqrt{\left(\pi\int_0^a rJ_n^2(a_{m,n}r)\,dr\right)}}, \text{ when } n > 0,$$

and

$$\phi_{m,0}(r, \theta) = \frac{J_0(a_{m,0}r)}{\sqrt{\left(2\pi\int_0^a rJ_0^2(a_{m,0}r)\,dr\right)}}.$$

The characteristic numbers $\lambda_{m,n}$ being $a_{m,n}^2$, where $a_{m,n}$ is the mth positive root of the equation $J_n(a\alpha)=0$.

Also
$$K(x,y\,;\,x',y')=\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{J_0(a_{m,0}r)J_0(a_{m,0}r')}{a_{m,0}^2\int_0^a rJ_0^2(a_{m,0}r)\,dr}$$
$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_n(a_{m,n}r)J_n(a_{m,n}r')}{a_{m,n}^2\int_0^a rJ_n^2(a_{m,n}r)\,dr}\cos n(\theta-\theta').$$

For a fuller discussion of these and other similar problems reference may be made to Kneser's book, cited above.

118. Three-Dimensional Problems.

The work of § 116 can be extended to three dimensions by using as Green's function the solution of $\nabla^2 u=0$, which vanishes on the surface of the solid, and is finite and continuous, as also its first and second differential coefficients, inside the solid, except at the point (x',y',z'), where it becomes infinite as $1/4\pi r$ when $r\to 0$. This function is the steady temperature due to a constant source of a certain strength at the point (x', y', z'), the surface of the solid being kept at temperature zero.

The equation
$$\frac{\partial v}{\partial t}=\kappa\nabla^2 v$$
is reduced to
$$\nabla^2\phi+\lambda\phi=0$$
by the substitution
$$v=e^{-\kappa\lambda t}\phi(x,y,z).$$
And we have
$$\phi(x,y,z)=\lambda\iiint K(x,y,z;\,x',y',z')\,\phi(x',y',z')\,dx'\,dy'\,dz',$$
the integral being taken through the solid.

EXAMPLES ON THE CONDUCTION OF HEAT.*

1. An infinite solid is bounded by the plane $x=0$, across which it radiates into a medium at zero. The solid is initially heated throughout to the uniform temperature v_0. Find the temperature at any point at any subsequent time, and prove that at the time t the temperature at the surface is given by

$$\frac{2v_0}{\sqrt{\pi}}\int_0^\infty e^{-u^2-2\lambda u\sqrt{(\kappa t)}}du.$$

2. If the temperature of an infinite solid has different uniform values V, V' on opposite sides of a given plane, prove that at any subsequent time the temperature is given by the expression,

$$\frac{V+V'}{2}+\frac{V-V'}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{(\kappa t)}}}e^{-\lambda^2}d\lambda,$$

x being measured from the plane towards the side where the temperature was initially V.

3. A uniform bar is given with the two halves of its length at two different temperatures v_0 and v_1. Find the temperature of any point of it at any subsequent time, the surface being so protected that there is no gain or loss of heat from without. For example, take an iron bar 50 cm. long. The thermal conductivity of iron (C.G.S. units) is ·16 (water being standard substance) and its thermal capacity per unit volume is ·875. Prove that at 1400 seconds from the beginning the temperature at either end will be

$$\tfrac{1}{2}(v_0+v_1)+\tfrac{1}{2}(v_0-v_1)\frac{4}{\pi}\left(\frac{1}{e}-\frac{1}{3e^9}+\frac{1}{5e^{25}}-\dots\right).$$

4. A bar of length l is heated so that its two ends are at the temperature zero. If initially the temperature is given by

$$v=\frac{cx(l-x)}{l^2},$$

show that the temperature at the time t at any point is given by

$$v=\frac{8ce^{-\lambda t}}{\pi^3}\left\{e^{-\frac{\pi^2\kappa t}{l^2}}\sin\frac{\pi x}{l}+\frac{1}{3}e^{-\frac{3^2\pi^2\kappa t}{l^2}}\sin\frac{3\pi x}{l}+\dots\right\}.$$

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* These problems are mostly taken from the Examination Papers for the Cambridge Mathematical Tripos. Some of them have already been published in Turner's *Examples on Heat and Electricity*.

5. One end of an infinite rod is kept for a long time at temperature v_0, there being surface radiation. A part whose extremities are distant b and $b+l$ from this end is then cut from the rod and kept from loss or gain of heat. Show that the temperature at time t at a point distance x from the end of the part is

$$v_0 e^{-ab}\frac{1-e^{-al}}{a}+v_0 a e^{-ab}\sum_{n=1}^{\infty}\frac{1-\cos n\pi e^{-al}}{a^2+\frac{n^2\pi^2}{l^2}}e^{-\kappa\frac{n^2\pi^2 t}{l^2}}\frac{\cos n\pi x}{l},$$

where a is a quantity depending upon the material of the rod.

6. One face $x=c$ of an infinite slab is kept at temperature zero. The temperature of the other, $x=0$, is kept up to the time τ at pt, p being a constant. After the time τ it is kept at a constant temperature. Find expressions for the temperature at any time, and show that if t is so great that $e^{-\beta^2(t-\tau)}$ may be neglected, the total quantity of heat which has passed across unit area of the surface up to the time t is

$$\frac{pK\tau}{2c}(2t-\tau)+\frac{pcs\tau}{3},$$

where s is the capacity for heat per unit volume, K is the conductivity, and

$$\beta^2=\frac{\pi^2 K}{c^2 s}.$$

7. A uniform cylindrical bar, of length l and small cross section, is kept at a constant temperature v_0 at one end and placed in a medium at temperature zero. If the temperature at a distance x from the end in the steady state is $v_0 e^{-ax}$, prove that half the radius of the bar and the ratio of the conductivity to the emissivity are each equal to a^{-1}.

When the steady state is attained the sides of the bar are coated with an adiathermanous substance, but its further end is left unaltered, the nearer end being still kept at the temperature v_0. Prove that the distribution of temperature at the time t is

$$v=v_0+\sum a_m \sin mx e^{-\kappa m^2 t},$$

where m is a root of the equation

$$a\tan ml+m=0$$

and
$$a_m=\frac{4mv_0}{2ml-\sin 2ml}\left\{\frac{\cos ml}{m}-\frac{a^2+me^{-al}(a\sin ml+m\cos ml)}{m(a^2+m^2)}\right\}.$$

8. If the surface of a semi-infinite solid has been subjected for an infinite time to the temperature $v=a+b\sin pt$, show that the distance from the surface at which the amplitude of the fluctuation of temperature is $\frac{1}{n}$ times that at the surface is $\sqrt{\left(\frac{2K}{p\rho c}\right)}\log n$.

9. A solid is bounded by the planes $x=0$ and $x=l$. Discuss the following cases, where the surface temperatures have been kept at the given

values so long that the distribution of temperature in the solid is purely periodic:

(i) $x=0$ at $v=a+b\sin pt$: $x=l$ at zero.

(ii) $x=0$ at $v=a+b\sin pt$: $x=l$, impervious to heat.

(iii) $x=0$ and $x=l$ at $v=a+b\sin pt$.

(iv) $x=0$ at $v=a+b\sin pt$: $x=l$ at $v=a-b\sin pt$.*

10. An infinite isotropic solid is bounded by an infinite plane and radiates across that plane into a medium at temperature $A\cos(\lambda t+\beta)$. Prove that after a time so great that all traces of the initial distribution of temperature throughout the solid have disappeared, the temperature at a distance x from the boundary is

$$\frac{hA}{\sqrt{(h^2+2h\mu+2\mu^2)}}e^{-\mu x}\cos(\lambda t-\mu x+\beta-\epsilon),$$

where
$$\epsilon=\tan^{-1}\frac{\mu}{\mu+h},\quad \mu^2=\frac{\lambda}{2\kappa},$$

and h, κ have the usual meanings.

Find the corresponding formula when the temperature of the medium is $f(t)$.

11. A uniform rod of length l, cross-section S, perimeter p, conductivity K and emissivity H, capacity for heat C, density D, and electrical resistance R, is placed in a medium at temperature zero, and has one end heated to temperature θ_0, the other end being kept at zero until the temperature is steady. An electrical current of strength I is now passed along the rod from the cold to the hot end. Show that when the temperature has again become steady, the rise of temperature due to the current at a point distant x from the cold end is

$$\theta=\frac{I^2RS}{Hp}+\frac{I\sigma\theta_0}{(HpKS)^{\frac{1}{2}}}\,\frac{\cosh x\left(\dfrac{Hp}{KS}\right)^{\frac{1}{2}}}{\sinh l\left(\dfrac{Hp}{KS}\right)^{\frac{1}{2}}},$$

where σ is the electrical conductivity of heat and $\sigma\dfrac{d\theta}{dx}$ is neglected.

12. Two uniform plates of the same substance and thickness a are in contact, and one slips over the other with constant velocity v, the friction per unit area being F. The outside surface of one is impervious to heat, and that of the other is kept at zero temperature. Show that at any time t their temperatures at a distance x from the impervious surface are given by

$$\theta=\frac{Fv}{JC}\left(a+\Sigma A_{2n+1}e^{-\frac{(2n+1)^2\pi^2C^2t}{16a^2c^2}}\cos(2n+1)\frac{\pi x}{4a}\right),$$

$$\theta=\frac{Fv}{JC}\left(2a-x+\Sigma A_{2n+1}e^{-\frac{(2n+1)^2\pi^2C^2t}{16a^2c^2}}\cos(2n+1)\frac{\pi x}{4a}\right),$$

where C is the conductivity, c is the thermal capacity per unit volume, and J is the mechanical equivalent of heat.

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* Cf. Kirsch, *Die Bewegung der Wärme in den Cylinderwandungen der Dampfmaschine*, p. 68, Leipzig, 1886.

13. An infinite homogeneous slab whose bounding planes are $x=\pm a$ is placed between two media, one beyond the plane $x=-a$, at which the temperature is $v_0 e^{-\kappa n^2 t}$, and the other beyond the plane $x=+a$, at which the temperature is zero. Show that if the ratio of the emissivity to the conductivity is $n\tan\beta$, and κ has the usual meaning, the temperature within the slab at time t is given by

$$v = v_0 e^{-\kappa n^2 t}\sin\beta \operatorname{cosec} 2(\beta - na)\cos(n(x-a)+\beta).$$

What would be the temperature at any point of the slab if the temperatures of the media at the two sides of the slab were $v_0 e^{-\kappa n_0^2 t}$ and $v_1 e^{-\kappa n_1^2 t}$?

14. A solid is bounded by two infinite parallel planes. Taking into account the radiation from its surfaces, show that the temperature at any internal point will be given by

$$\Sigma A e^{-\kappa^2 m^2 t}\cos\left(\frac{\lambda(x+a)+\lambda'(x-a)}{2a}\right),$$

where $$\lambda=\tan^{-1}\frac{l}{m},\quad \lambda'=\tan^{-1}\frac{l'}{m},$$

l, l' are the emissivities of the two faces, supposed unequal, and $2a$ is the thickness of the solid. The origin is taken midway between the faces, and m is determined by the equation

$$ma=\frac{n\pi+\lambda+\lambda'}{2},$$

n being any integer.

15. A ring of uniform small section so coated as to have everywhere the same emissivity is made partly of brass and partly of iron, and one of the junctions is kept at a constant temperature, while the whole cools in air. Determine the ratio of the lengths of the two parts when the coolest place is at the other junction.

16. A thin circular ring whose surface is impermeable to heat is heated by a continuous source of heat of strength Q. Show that the temperature at time t is given by

$$v=\frac{Qt}{2\pi cA\sigma}+\frac{Qc(\phi-\pi)^2}{4\pi AK}-\frac{\pi Qc}{12AK}-\frac{Qc}{\pi AK}\sum_1^\infty\frac{1}{n^2}\cos n\phi\, e^{-\lambda t},$$

where
K = the conductivity,
σ = the thermal capacity per unit volume,
A = the cross section of ring,
c = its mean radius,
s = an arc of the ring measured from the heated spot,
ϕ = the angle subtended by s at the centre,

and $$\lambda\sigma=\frac{Kn^2}{c^2}.\text{*}$$

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* Cf. Niven, *London, Proc. R. Soc.* (A), 76, p. 42, 1905.

17. Any point, the radius through which makes an angle θ with a fixed radius of the edge of a circular disc of radius a, is maintained at temperature $f(\theta)$, where $f(\pi+\theta)=-f(\theta)$. Show that when the motion of heat is steady the temperature at (r, θ) is

$$\frac{2}{\pi}\int_0^\pi f(\theta+\phi)\frac{ar(a^2-r^2)\cos\phi}{a^4-2a^2r^2\cos 2\phi+r^4}d\phi,$$

the disc being supposed not to radiate heat.

18. If the diameter of a circle be kept at the temperature v_1, and the circumference at temperature v_2, prove that the temperature at any point is

$$v_1+\frac{2}{\pi}(v_2-v_1)\tan^{-1}\frac{2ay}{x^2+y^2-a^2},$$

and that the isothermal lines are circles passing through the ends of the diameter which is taken as axis of x, a being the radius of the circle.

19. A plate extends to infinity in two directions and is bounded by two straight edges which meet at right angles in A. Both edges are at temperature zero, except a portion AB of one edge, which is kept at temperature unity. Prove that the temperature at any point P is

$$\frac{2}{\pi}\tan^{-1}\left(\frac{a_2b_2}{a_1b_1}\right) \quad \text{or} \quad \frac{1}{\pi}(\angle APB-\angle APC),$$

where a_1, b_1 are the semi-axes of the ellipse and a_2, b_2 of the hyperbola, which can be drawn through P, having A as centre and B as focus, and where C ies in BA produced, so that $AC=AB$.

20. If in a sector of radius a and angle α the radii be maintained at the temperature v_1 and the circumference at the temperature v_2, verify that the temperature at any point of the section will be

$$\frac{2v_1}{\pi}\tan^{-1}\left(\frac{r^{\frac{2\pi}{\alpha}}-a^{\frac{2\pi}{\alpha}}}{2a^{\frac{\pi}{\alpha}}r^{\frac{\pi}{\alpha}}\sin\frac{\pi\theta}{\alpha}}\right)+\frac{2v_2}{\pi}\tan^{-1}\left(\frac{2a^{\frac{\pi}{\alpha}}r^{\frac{\pi}{\alpha}}\sin\frac{\pi\theta}{\alpha}}{r^{\frac{2\pi}{\alpha}}-a^{\frac{2\pi}{\alpha}}}\right).$$

21. A plane area is bounded by a semi-ellipse and its axis major. The elliptic boundary is maintained at the uniform temperature unity and the straight boundary at the temperature zero. Prove that the temperature at any point within the area is

$$\frac{4}{\pi}\left(\frac{\sinh\phi}{\sinh\alpha}\sin\theta+\frac{1}{3}\frac{\sinh 3\phi}{\sinh 3\alpha}\sin 3\theta+\dots\right),$$

where $c\cosh\phi$, $c\sinh\phi$; $c\cos\theta$, $c\sin\theta$ are the semi-axes of the ellipse and hyperbola through the given point, confocal with the boundary, and α is the value of ϕ at the boundary.

22. A circular cylinder of infinite length is divided into four compartments by planes through the axes at right angles to each other. Measuring

θ from one of the planes, the temperatures of the successive quadrants of the surface are maintained at the respective values

$$T\sin\theta,\quad T\cos\theta,\quad T\sin\theta,\quad T\cos\theta.$$

If v is the temperature at a point inside, and if the radius of the cylinder is the unit of length, prove that

$$\frac{\pi v}{T}=\frac{\pi T}{2}(\sin\theta+\cos\theta)+r(\cos\theta-\sin\theta)-r^3(\cos 3\theta+\sin 3\theta)$$

$$+\frac{r^5}{3}(\cos 5\theta-\sin 5\theta)-\frac{r^7}{3}(\cos 7\theta+\sin 7\theta)$$

$$+\frac{r^9}{5}(\cos 9\theta-\sin 9\theta)-\frac{r^{11}}{5}(\cos 11\theta+\sin 11\theta)$$

$$+\text{etc.}$$

23. If a slit be made in the plane along the line $\theta=0$, commencing at the origin and extending indefinitely in the positive direction, and if both sides of the slit be maintained at zero temperature during the diffusion, prove that

$$v=\frac{q}{4\pi\kappa t}\int_0^\infty e^{-\frac{r^2+r'^2}{4\kappa t}-a^2}\left\{e^{2a\sqrt{\left(\frac{rr'}{\kappa t}\right)}\cos\frac{1}{2}(\theta-\theta')}-e^{2a\sqrt{\left(\frac{rr'}{\kappa t}\right)}\cos\frac{1}{2}(\theta+\theta')}\right\}da$$

is the temperature at time t due to a line source of strength q generated at $t=0$ at (r', θ').

Show also how to obtain the corresponding expression for the portion of the infinite plane bounded by two straight edges inclined at an angle $2\pi/(2m+1)$, m being integral, the edges being both maintained at zero.

24. A conducting sphere initially at zero temperature has its surface kept at a constant temperature c for a given time, after which it is kept at zero. Find the temperature at any time in the second stage.

25. A sphere of radius a with initial temperature v_0 is surrounded by an infinite medium of the same material as the sphere and of initial temperature zero. Prove that the temperature at distance r from the centre of the sphere at the time t is given by

$$v=\frac{v_0}{\sqrt{\pi}}\left[\int_{\frac{r-a}{2\sqrt{(\kappa t)}}}^{\frac{r+a}{2\sqrt{(\kappa t)}}}e^{-\lambda^2}d\lambda-\frac{\sqrt{(\kappa t)}}{r}\left\{e^{-\frac{(r-a)^2}{4\kappa t}}-e^{-\frac{(r+a)^2}{4\kappa t}}\right\}\right].$$

26. A uniform sphere of radius a is at a uniform temperature v_0, and is surrounded by a spherical shell of thickness a at zero. The whole is left to cool in a medium at zero. Prove that

$$v=v_0\sum_\sigma\frac{4}{\sigma}\,\frac{\sin\sigma a-a\sigma\cos\sigma a}{4\sigma a-\sin 4\sigma a}\,\frac{\sin\sigma r}{r}e^{-\kappa\sigma^2 t},$$

where the values of σ are given by

$$\tan 2\sigma a=\frac{2\sigma a}{1-2ha}.$$

Also consider the case when the two substances have different conductivities.

27. A homogeneous solid bounded by two concentric spheres of radii a and $2a$ respectively has its inner surface coated with a layer of a substance impervious to heat. The solid is raised to the temperature V_0 and left to cool in a medium at zero temperature.

Prove that
$$v=\sum_\lambda A_\lambda e^{-\kappa\lambda^2 t}\frac{\lambda a\cos\lambda(r-a)+\sin\lambda(r-a)}{r},$$
where λ is a root of
$$(1-2ah+2\lambda^2a^2)\sin\lambda a=(1+2ah)\lambda a\cos\lambda a.$$
Show how A_λ may be found.

28. A sphere of radius c is symmetrically heated so that its initial temperature at a distance r from the centre is $f(r)$. It is then allowed to cool by radiation into a medium at zero. Prove that if the sphere is very small so that powers of hc above the first can be neglected in comparison with unity, except just at the beginning of the cooling, the temperature becomes approximately proportional to
$$\frac{1}{hr}e^{-\frac{3\kappa ht}{c}}\sin r\sqrt{\left(\frac{3h}{c}\right)}.$$

29. The initial temperature at a point of a sphere of radius c at a distance r from the centre is
$$\frac{\sinh\left(\kappa r-\dfrac{r}{c}\right)}{r},$$
and the sphere is surrounded by a medium at temperature zero. If h is the ratio of the emissivity and the conductivity and $\lambda_1, \lambda_2, \ldots$ are the roots of the equation
$$c\lambda\cos c\lambda+(hc-1)\sin c\lambda=0,$$
prove that the subsequent temperature at the time t within the sphere is equal to
$$\frac{2(hc-1)}{r}\sum_1^\infty e^{hc-1-\kappa\lambda_m^2 t}\frac{\sin\lambda_m r\sin\lambda_m c}{\lambda_m^2c^2+hc(hc-1)}.$$

30. A sphere of radius c has initial temperature
$$\frac{A}{2r}e^{1-\frac{r}{c}},$$
and radiation takes place at its surface into a medium at zero. Show that if λ_m is a root of the equation
$$c\lambda\cos c\lambda+(hc-1)\sin c\lambda=0,$$
the temperature at the time t is given by
$$\frac{A}{r}\sum_1^\infty\frac{P}{Q}\sin\lambda_m r\frac{(hc-2)\sin\lambda_m c+ec\lambda_m}{\lambda_m^2c^2+1}e^{-\kappa\lambda_m^2 t},$$
where
$$P=c^2\lambda_m^2+(hc-1)^2,$$
$$Q=c^2\lambda_m^2+hc(hc-1).$$

31. The initial temperature at any point of a sphere exposed in an infinite medium at temperature zero is given by
$$\frac{1}{r}Vc\sin\frac{\pi r}{2c}e^{1-\frac{r}{c}},$$

c being the radius and r the distance of the point from the centre. Show that, if λ is a root of the equation

$$c\lambda \cos c\lambda + (hc-1)\sin c\lambda = 0,$$

the temperature at the same point after a time t is

$$\frac{1}{r}\sum_1^\infty B_m e^{-\kappa\lambda_m^2 t}\sin\lambda_m r,$$

where

$$B_m = \frac{V\pi c\sqrt{\{\lambda_m^2c^2+(hc-1)^2\}}\left[(hc-1)\left(1+\frac{\pi^2}{4}\right)-\lambda_m^2c^2(hc-3)+2c\lambda_m c\sqrt{\{\lambda_m^2c^2+(hc-}\right.}{(\lambda_m^2c^2+hc(hc-1))\left\{1+\left(\frac{\pi}{2}-\lambda_m c\right)^2\right\}\left\{1+\left(\frac{\pi}{2}+\lambda_m c\right)^2\right\}}$$

32. A homogeneous sphere of radius a is heated so that the initial temperature is $v_0 = \frac{A}{r^3}(\cos\theta \sin mr - mr\cos mr)$.

Show that the temperature at time t is $v_0 e^{-\kappa m^2 t}$, where m is a root of the equation

$$(ah-2)(ma\cot ma - 1) = m^2a^2,$$

the external medium being at zero temperature.

33. A solid globe of metal, radius a, conductivity κ_1, thermal capacity per unit volume c_1, is surrounded by one of another metal, outer radius b, conductivity and capacity κ_2 and c_2, the whole radiating into a medium at zero. Prove that, if at any time the temperatures of the two metals are represented by

$$u = \frac{\sin mr}{mr},$$

$$v = \frac{\sin ma\cos n(r-a)}{mr} + \frac{\kappa_1}{\kappa_2}\frac{m}{a}\frac{\cos ma\sin n(r-a)}{mn} - \left(\frac{\kappa_1}{\kappa_2}-1\right)\frac{\sin ma\sin n(r-a)}{mnar},$$

where

$$m^2\frac{\kappa_1}{c_1} = n^2\frac{\kappa_2}{c_2} = \theta,$$

the temperatures at any subsequent time will be

$$e^{-\theta t}u, \quad e^{-\theta t}v;$$

and determine the equation connecting n with the emissivity of the outer metal.

34. A solid sphere is surrounded by a concentric non-conducting spherical surface, and the space between is filled with hot liquid kept at a uniform temperature by agitation. Prove that if v is the temperature at a point in the sphere, and v', the temperature of the liquid at any time, then v, v' are of the forms

$$v = c + \Sigma A_n \frac{\sin nr}{r} e^{-\kappa n^2 t},$$

$$v' = c + \Sigma A_n \frac{m^2}{m^2-n^2}\frac{\sin na}{a} e^{-\kappa n^2 t},$$

where $\frac{am^2}{3h}$ is the ratio of the heat capacity of the sphere to that of the liquid, and x is any root of the equation

$$xa=\left(1+ha\frac{x^2}{m^2-x^2}\right)\tan xa.$$

35. In an infinite conductor made of uniform material an instantaneous spherical source of strength Q is generated over the surface of a sphere of radius a and left to diffuse through the conductor. Prove that if the heat generated over equal elements of the spherical surface is everywhere the same, the subsequent temperature at a point distant r from the centre of the sphere is

$$\frac{Q}{\sqrt{(\pi\kappa t)}}\,\frac{1}{4\pi ar}\,e^{-\frac{r^2+a^2}{4\kappa t}}\sinh\frac{ar}{2\kappa t}.$$

36. If over every element dS of a spherical surface in an infinite solid sources of strength $Y_n(x', y', z')dS$ of heat are generated ($Y_n(x', y', z')$ being a solid spherical harmonic of degree n; x', y', z' being the coordinates of the element dS referred to axes through the centre), prove that, at the point (x, y, z) distant r from the centre at time t,

$$v=\frac{a^{2n+2}}{2\pi^{\frac{1}{2}}(\kappa t)^{n+\frac{3}{2}}}Y_n(x, y, z)e^{-\frac{r^2+a^2}{4\kappa t}}\frac{d^n}{d(u^2)^n}\left(\frac{\sinh u}{u}\right),$$

where $$u=\frac{ar}{2\kappa t}.$$

37. The surface temperature of a sphere of thermometric conductivity κ is made to vary according to the law $v=S_n\cos ct$, where S_n is a surface harmonic of degree n. Prove that the consequent fluctuation of temperature in the interior is given by the formula

$$v=\frac{[P(r)P(a)+Q(r)Q(a)]\cos\sigma t+[P(r)Q(a)-Q(r)P(a)]\sin\sigma t}{(P(a))^2+(Q(a))^2}\left(\frac{r}{a}\right)^n S_n,$$

where $$P(r)=1-\frac{\left(\frac{\sigma}{\kappa}\right)^2 r^4}{2\,.\,4\,.\,2n+3\,.\,2n+5}+\frac{\left(\frac{\sigma}{\kappa}\right)^4 r^8}{2\,.\,4\,.\,6\,.\,8\,.\,2n+3\ldots 2n+9}-\ldots,$$

$$Q(r)=\frac{\left(\frac{\sigma}{\kappa}\right)r^2}{2\,.\,2n+3}-\frac{\left(\frac{\sigma}{\kappa}\right)^3 r^6}{2\,.\,4\,.\,6\,.\,2n+3\ldots 2n+7}+\ldots.$$

Examine and interpret the forms which the result assumes (1) when σ is large compared with $\frac{\kappa}{a^2}$, and (2) when n is large compared with $\sigma\frac{a^2}{\kappa}$.

38. A homogeneous solid sphere has half its surface, viz. the portion bounded by a great circle, maintained at temperature unity, while the other half is maintained at zero. Find the steady motion of heat. Show also that the mean of the temperatures at two points on the same diameter and equally distant from the centre is the same for all diameters.

Let the sphere while in this state of temperature be enclosed in a closely fitting envelope impervious to heat and left to itself. Show how to find the temperature at any point after any time t, and when t is very great, show that the temperature at a point whose coordinates are (r, θ) is approximately

$$\tfrac{1}{2} + A(\sigma r \cos \sigma r - \sin \sigma r)\frac{\cos\theta}{r^2} e^{-\kappa\sigma^2 t},$$

where σ is a constant depending on the radius of the sphere. Show how to determine this constant.

39. A line source of strength Q is instantaneously generated along the axis of an infinitely long circular cylinder at the time $t=0$. The temperature v was everywhere previously zero, and the temperature of the boundary $r=a$ is maintained at zero. Prove that at any subsequent instant

$$v = \frac{Q}{\pi a^2} \sum_s \frac{J_0(m_s r)}{(J_1(m_s a))^2} e^{-\kappa m_s^2 t},$$

where κ is the thermometric conductivity, and the quantities m_s are the positive roots of the equation

$$J_0(m_s a) = 0.$$

Prove that when a is made infinite the above expression assumes the form

$$v = \frac{Q}{2\pi} \int_0^\infty J_0(ru) u e^{-\kappa u^2 t} du,$$

and by comparison with an independent solution of the problem, evaluate the definite integral.

APPENDIX I

NOTE ON BESSEL'S FUNCTION.

1. The Bessel's function $J_n(z)$ is given by the equation

$$J_n(z)=\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{n+2r}}{\Gamma(r+1)\Gamma(n+r+1)},$$

and is thus defined for all values of n, when the general Gamma Function* is used. $J_n(z)$, when n is not an integer, will be many-valued, but it will be made single-valued by restricting the complex variable to a complete revolution: *e.g.* by taking $|\arg z|<\pi$.

For other values of the argument we use the equation

$$J_n(ze^{r i\pi})=e^{nr i\pi}J_n(z).$$

2. When n is not an integer, $J_n(z)$ and $J_{-n}(z)$ are independent solutions of Bessel's equation, but $J_n(z)=(-1)^n J_{-n}(z)$, when n is integral.

For a second solution, available for all values of n, we choose

$$Y_n(z)=\frac{J_n(z)\cos n\pi-J_{-n}(z)}{\sin n\pi},$$

the limit being taken when n is an integer.

It is known† that with this definition $Y_n(z)$ is infinite when $z=0$, and that $Y_0(z)$ and $Y_n(z)$ are given by

$$\pi Y_0(z)=2J_0(z)(\log(z/2)+\gamma)+(z/2)^2-\frac{1+1/2}{(2\,!)^2}(z/2)^4+\frac{1+1/2+1/3}{(3\,!)^2}(z/2)^6-\dots,$$

$$\pi Y_n(z)=J_n(z)\left\{2\log(z/2)+2\gamma-\sum_{m=1}^{n+r}m^{-1}-\sum_{m=1}^{r}m^{-1}\right\}-\sum_{r=0}^{n-1}(z/2)^{-n+2r}\frac{(n-r-1)!}{r!}.$$

3. The Bessel's function of the second kind defined above and denoted by $Y_n(z)$ was first used by Weber.‡ It is taken as the standard function of the second kind by Nielsen,§ but he uses the notation $Y^n(z)$ and calls it Neumann's function. As a matter of fact, Neumann (K.) in his *Theorie der Bessel'schen Functionen* (Leipzig, 1867) was concerned only with positive

* Whittaker and Watson, *loc. cit.* (3rd Ed.), Ch. XII.

† Whittaker and Watson, *loc. cit.* (3rd Ed.), p. 372; Watson, *loc. cit.*, §§3. 51 (3), 3. 52 (3).

‡ *Math. Ann.*, *Leipzig*, 6, p. 148, 1873.

§ Nielsen, *loc. cit.*, p. 10.

integral values of n, and his function of the second kind, for which he used the notation $Y^n(z)$, is connected with our $Y_n(z)$ by the relation

$$Y^n(z)=\frac{\pi}{2}Y_n(z)+J_n(z)(\log 2-\gamma),$$

where γ is Euler's Constant,* namely $\underset{n\to\infty}{Lt}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)$.

The function we denote by $Y_n(z)$ is the same as Gray and Mathews' $\Upsilon_n(z)$.† It should be noticed that Schafheitlin's $Y_n(z)$ is minus ours:‡ also that Gray and Mathews' $Y_n(z)$ is Neumann's function $Y^n(z)$.

4. In many questions the functions denoted by $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are very useful.§ They are defined by the simple relations

$$H_n^{(1)}(z)=J_n(z)+iY_n(z),\quad\text{and}\quad H_n^{(2)}(z)=J_n(z)-iY_n(z).$$

They may be described as Bessel's functions of the third kind, and they must find a permanent place in the treatment of Bessel's functions because of the simple formulae for $|\arg z|<\pi$:

$$H_n^{(1)}(z)=\sqrt{\left(\frac{2}{\pi z}\right)}e^{i\left(z-(2n+1)\frac{\pi}{4}\right)},$$

approximately in the upper part of the z-plane, when $|z|$ is large;

$$H_n^{(2)}(z)=\sqrt{\left(\frac{2}{\pi z}\right)}e^{-i\left(z-(2n+1)\frac{\pi}{4}\right)},$$

approximately in the lower part of the z-plane, when $|z|$ is large.‖

5. Since, in the upper part of the z-plane when $|z|$ is large, we have the approximation**

$$J_n(z)=\frac{1}{\sqrt{(2\pi z)}}e^{-i\left(z-(2n+1)\frac{\pi}{4}\right)},$$

it follows that at infinity in the upper part of the z-plane $J_n(z)$ is infinite, $H_n^{(1)}(z)$ vanishes, and $J_n(az)H_n^{(1)}(bz)$ vanishes, when a and b are real and positive, $a<b$.

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* Cf. Whittaker and Watson, *loc. cit.* (3rd Ed.), p. 235.

† Gray and Mathews, *loc. cit.*, p. 64, (131).

‡ Schafheitlin, *Theorie der Besselschen Funktionen*, p. 44, Leipzig, 1908.

§ Nielsen, *loc. cit.*, p. 16. Watson, *loc. cit.*, § 3. 6 (1).

‖ Cf. Whittaker and Watson, *loc. cit.* (3d Ed.), pp. 368, 371; Watson, *loc. cit.*, § 7. 2 (1).

** Whittaker and Watson, *loc. cit.*, (3rd Ed.), p. 368; Watson, *loc. cit.*, § 7. 21 (1).

APPENDIX II

BIBLIOGRAPHY*

The Conduction of Heat

TREATISES.

Fourier. Théorie analytique de la chaleur. Paris, 1822.
English translation (Freeman), Cambridge, 1872. German translation (Weinstein), Berlin, 1884.

Poisson. Théorie mathématique de la chaleur. Paris, 1835. Supplément, Paris, 1837.

Lamé. Leçons sur les fonctions inverses des transcendantes et les surfaces isothermes. Paris, 1857.

Riemann. Partielle Differentialgleichungen und deren Anwendungen auf physikalischen Fragen. Braunschweig, 1869. [See also Weber, *Die partiellen Differential-gleichungen der mathematischen Physik*, (2 Aufl.), Braunschweig, 1910-12. This work is also known as the fifth edition of Riemann's lectures referred to above. It is cited in the text as Weber-Riemann.]

Dronke. Einleitung in die analytische Theorie der Wärmeverbreitung. (Nach. A. Beer u. J. Plücker.) Leipzig, 1882.

Kirsch. Die Bewegung der Wärme in den Cylinderwandungen der Dampfmaschine. Leipzig, 1886.

D'Amicis. Introduzione alla teoria matematica della propagazione del calore. Torino, 1891.

Poincaré. Théorie analytique de la propagation de la chaleur. Paris, 1895.

Weber. See above under Riemann. 1900-1.

Boussinesq. Théorie analytique de la chaleur. Paris, 1901-3.

Burkhardt. Entwicklungen nach oscillirenden Functionen. Jahresber. D. Math. Ver., Leipzig, Bd. X., Heft. II., 1901.

Carslaw. Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat. London, 1906.

---

* The abbreviated titles for the Journals are taken from the *International Catalogue of Scientific Literature.*

INGERSOLL AND ZOBELL. The Mathematical Theory of Heat Conduction. Boston, 1913.

In many other works considerable space is given to the Mathematical Theory of the Conduction of Heat. The following may be specially mentioned :—

BIOT. Traité de Physique, T. IV. Paris, 1814.

HEINE. Theorie der Kugelfunctionen und der verwandten Functionen, Bd. II. (2 Aufl.). Berlin, 1881.

HELMHOLTZ. Vorlesungen über theoretische Physik, Bd. VI. Theorie der Wärme (herausgegeben von Richarz), Leipzig, 1903.

KIRCHHOFF. Vorlesungen über mathematischen Physik, Bd. IV. Theorie der Wärme. Leipzig, 1891.

MATHIEU. Cours de Physique Mathématique. Paris, 1873.

POCKELS. Über die Differentialgleichung $\Delta u+\kappa^2 u=0$ und deren Auftreten in der mathematischen Physik. Leipzig, 1891.

Also the articles on "Heat" in the *Enc. Brit.* (9th Ed.), 1878, by Kelvin, and *Supplement*, 1902, by Callendar, and by Sommerfeld in *Enc. d. math. Wiss.*, Bd. II., 1904, entitled "Randwerthaufgaben in die Theorie der partiellen Differentialgleichungen."

MEMOIRS AND PAPERS.

FOURIER. (See *F.S.*, p. 303.)

POISSON. (i) Sur la distribution de la chaleur dans les corps solides. (Three papers.) J. éc. polytech., Paris, 12, 1823.

(ii) Sur la distribution de la chaleur dans un anneau homogène et d'une épaisseur constante, lorsque la température du lieu où il est placé varie d'un point à un autre. Connaissance des Temps, 1826.

(iii) Sur la température des différents points de la terre particulièrement près de sa surface. Connaissance des Temps, 1827.

PAGANI. (i) Sur l'intégration complète de l'équation du mouvement de la chaleur dans une barre prismatique d'une petite épaisseur. Quetelet, Correspondance Mathématique, 3, 1827.

(ii) Sur le mouvement de la chaleur dans une sphère composée de deux parties hétérogènes et concentriques. Quetelet, Correspondance Mathématique, 4, 1828.

DIRICHLET. Solution d'une question relative à la théorie mathématique de la chaleur. J. Math., Berlin, 5, 1830.

LIBRI. Sur la théorie de la chaleur. J. Math., Berlin, 7, 1831.

DUHAMEL. (i) Sur les équations générales de la propagation de la chaleur dans les corps solides dont la conducibilité n'est pas la même dans tous les sens. J. éc. polytech., Paris, 13, 1832.

(ii) Sur la méthode générale relative au mouvement de la chaleur dans les corps solides plongés dans les milieux dont la température varie avec les temps. J. éc. polytech., Paris, 14, 1833.

LAMÉ. (i) Sur la propagation de la chaleur dans les polyèdres principalement dans le prisme triangulaire régulier. J. éc. polyte Paris, 14, 1833.

(ii) Sur les surfaces isothermes dans les corps solides en équilibre température. Ann. chim. phys., Paris, 53, 1833.

POISSON. Théorie mathématique de la chaleur. J. Math., Berlin, 1834.

LAMÉ. Sur l'équilibre des températures dans les corps solides de fo cylindrique. J. math., Paris, 1, 1836.

LIOUVILLE. Sur le développement des fonctions ou parties des foncti en séries dont les divers termes sont assujettis à satisfaire à équation différentielle du second ordre contenant un param variable. J. math., Paris, 1, 1836. (Also 2, 1837 and 3, 1838

STURM. Sur les équations différentielles linéaires du second ordre. math., Paris, 1, 1836.

LAMÉ. Sur les surfaces isothermes dans les corps solides homog en équilibre de température. J. math., Paris, 2, 1837.

DUHAMEL. Sur les phénomènes thermo-mécaniques. J. éc. polyte Paris, 15, 1837.

LIOUVILLE ET STURM. Extrait d'un mémoire sur le développement fonctions en séries dont les differents termes sont assujettis à sa faire à une équation différentielle linéaire, contenant un param variable. J. math., Paris, 2, 1837.

POISSON. Sur les températures de la partie solide du globe, de l'at sphère, et du lieu de l'espace où la Terre se trouve actuellem Paris, C.R. Acad. sci., 4, 1837.

LAMÉ. (i) Sur les surfaces isothermes dans les corps solides homog en équilibre de température. Paris, Mém. Sav. Étrangers, 5, 1

(ii) Sur la propagation de la chaleur en polyèdres. Paris, Mém. Étrangers, 5, 1838.

LIOUVILLE. Observations sur un mémoire de M. Libri relatif à la th de la chaleur. J. math., Paris, 3, 1838.

DUHAMEL. Sur les surfaces isothermes dans les corps solides don conducibilité n'est pas la même dans tous les sens. J. m Paris, 4, 1839.

LAMÉ. (i) Sur l'équilibre des températures dans un ellipsoïde homo et solide. Paris, C.R. Acad. sci., 8, 1839.

(ii) Sur les axes des surfaces isothermes du second degré consid comme des fonctions de la température. J. math., Paris, 4, 1

(iii) Sur l'équilibre des températures dans un ellipsoïde à trois inégaux. J. math., Paris, 4, 1839.

(iv) Sur l'équilibre des températures dans les corps solides homog de forme ellipsoïdale, concernant particulièrement les ellipsoïde révolution. J. math., Paris, 4, 1839.

PECLET. Sur la détermination des coefficients de conducibilité métaux par la chaleur. Ann. chim. phys., Paris (Sér. 3), 2, 18

HEINE. Über einige Aufgaben, welche auf partielle Differentialgleichungen führen. J. Math., Berlin, 26, 1843.

KELVIN. (i) Note on a passage in Fourier's Heat. Cambridge and Dublin Math. Journal, 3, 1843.

(ii) On the Uniform Motion of Heat in homogeneous Solid Bodies and its Connection with the Mathematical Theory of Electricity. Cambridge and Dublin Math. Journal, 3, 1843.

(iii) On the Linear Motion of Heat. Cambridge and Dublin Math. Journal, 3, 1843.

(iv) Note on Orthogonal Isothermal Surfaces. Cambridge and Dublin Math. Journal, 3, 1843. (Cf. also 4, 1844.)

LAMÉ. (i) Sur les surfaces orthogonales et isothermes. J. math., Paris, 8, 1843.

(ii) Sur la méthode de recherche des surfaces isothermes. J. math., Paris, 8, 1843.

KELVIN. (i) On the Equations of the Motion of Heat referred to Curvilinear Coordinates. Cambridge and Dublin Math. Journal, 4, 1844.

(ii) Note on some points in the Theory of Heat. Cambridge and Dublin Math. Journal, 4, 1844.

HEARN. On the permanent state of heat in a thin uniform wire of any form, acted on by two sources of heat of equal intensity at its extremities. Phil. Mag., London (Ser. 3), 29, 1846.

LIOUVILLE. Lettres sur diverses questions d'analyse et de physique mathématique. J. math., Paris, 11, 1846.

STOKES. On the Critical Values of the Sums of Periodic Series. Cambridge, Trans. Phil. Soc., 8, 1847.

BONNET. Sur quelques cas particuliers d'équilibre de température dans les corps dont la conductibilité varie avec la position et direction. Paris, C.R. Acad. sci., 27, 1848.

DUHAMEL. Sur la propagation de la chaleur dans les crystaux. Paris, J. éc. polytech., Paris, 19, 1848.

LIOUVILLE. Sur l'équation aux différences partielles qui concerne l'équilibre de la chaleur dans un corps héterogène. J. math., Paris, 13, 1848.

AMSLER. Zur Theorie der Anziehung und der Wärme. J. Math., Berlin, 42, 1851.

STOKES. On the Conduction of Heat in Crystals. Cambridge and Dublin Math. Journal, 6, 1851.

AMSLER. Über die Gesetze der Wärmeleitung im Innern fester Körper; unter Berücksichtigung der durch ungleichförmige Erwärmung erzeugten Spannung. J. Math., Berlin, 42, 1852.

ÅNGSTRÖM. Neue Methode das Wärmeleitungsvermögen der Körper zu bestimmen. Ann. Physik, Leipzig, 114, 1861.

EVERETT. On a Method of reducing Observations of Underground Temperature, with Applications to the Monthly Mean Temperatures

of Underground Thermometers at the Royal Observatory, Edinburgh. Edinburgh, Trans. R. Soc., **22**, 1861.

KELVIN. On the Reduction of Observations on Underground Temperature, with Applications to Professor Forbes' Edinburgh Observations and the continued Calton Hill Series. Edinburgh, Trans. R. Soc., **22**, 1861.

MINNIGERODE. Über die Wärmeleitung in Krystallen. Diss. Göttingen, 1862.

NEUMANN, F. Expériences sur la conducibilité calorifique des solides. Ann. chim. phys., Paris, **66**, 1862.

NEUMANN, C. Über das Gleichgewicht der Wärme und das der Elektricität in einem Körper welcher von zwei nicht concentrischen Kugelflächen begrenzt wird. J. Math., Berlin, **62**, 1863.

ÅNGSTRÖM. Nachtrag zu dem Aufsatze; Neue Methode das Wärmeleitungsvermögen der Körper zu bestimmen. Ann. Physik, Leipzig, **123**, 1864.

EVERETT. Investigation of an Expression for the Mean Temperature of a Stratum of Soil in terms of the Time of Year. Edinburgh, Trans. R. Soc., **23**, 1864.

FORBES. An Experimental Enquiry into the Laws of the Conduction of Heat in Bars. Pt. I. Edinburgh, Trans. R. Soc., **23**, 1864.

KELVIN. On the Secular Cooling of the Earth. Edinburgh, Trans. R. Soc., **23**, 1864.

BETTI. Sopra la determinazione delle temperature variabili d'un cilindro. Pisa, Ann. delle Università Toscane (Ser. 2), **10**, 1867.

BOUSSINESQ. Sur un nouvel ellipsoïde qui joue un grand rôle dans la théorie de la chaleur. Paris, C.R. Acad. sci., **65**, 1867.

FORBES. An Experimental Enquiry into the Laws of the Conduction of Heat in Bars. Pt. II. Edinburgh, Trans. R. Soc., **24**, 1867.

BETTI. (i) Sopra la determinazione delle temperature di una lastra terminata. Ann. mat., Milano (Ser. 2), **1**, 1868.

(ii.) Sopra la determinazione delle temperature variabili di una lastra terminata, quando la conducibilitá non è la stessa in tutte le direzioni. Nuovo Cimento, Pisa, **28**, 1868.

(iii) Sopra la determinazione delle temperature nei corpi solidi omogenei. Firenze, Mem. della Soc. delle Sci. (Ser. 3), **1**, 1868.

CHRISTOFFEL. Sul problema delle temperature stazionarie e la rappresentazione di una data superficie. Ann. mat., Milano (Ser. 2), **1**, 1868.

FROSCH. Über den Temperaturzustand eines von zwei nicht concentrischen Kugelflächen eingeschlossen Körpers. Zs. Math., Leipzig, **13**, 1868. (Cf. also **17**, 1872.)

BOUSSINESQ. Sur les surfaces isothermes et sur les courants de chaleur dans les milieux homogènes chauffés en un de leur points. J. math., Paris (Sér. 2), **14**, 1869.

MATHIEU. Sur le mouvement de la température dans les corps renfermés entre deux cylindres circulaires excentriques et dans les cylindres lemniscatiques. J. math., Paris (Sér. 2), 14, 1869.

WEBER, H. Über die Integration der partiellen Differential-gleichung $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0$. Math. Ann., Leipzig, 1, 1869.

SCHLÄFLI. Über die partiellen Differential-gleichung $\frac{\partial w}{\partial t}=\kappa\frac{\partial^2 w}{\partial x^2}$. J. Math., Berlin, 72, 1870.

VON DER MÜHLL. Über den stationären Temperaturzustand. Math. Ann., Leipzig, 2, 1870.

KNAKE. Über die lineare Wärmebewegung in einem von zwei parallelen Wänden begrenzten Körper dessen Begrenzungen mit einem Gase in Berührung stehen. Diss. Halle, 1871.

WEBER, H. F. Über ein Problem der Wärmetheorie. Zurich, Vierteljahrsch. Natf. Ges., 16, 1871.

MATHIEU. Sur l'intégration des équations aux différences partielles de la physique mathématique. J. math., Paris (Sér. 2), 17, 1872.

WEBER, H. Über das Wärmeleitungsvermögen von Eisen und Neusilber. Ann. Physik, Leipzig, 146, 1872.

BAER. Über die Bewegung der Wärme in einer homogenen Kugel. Diss. Halle, 1875.

FORBES. On the Thermal Conductivity of Ice, and a New Method of Determining the Conductivity of Different Substances. Edinburgh, Proc. R. Soc., 8, 1875.

LANGER. Über die Wärmeleitung in einer homogenen Kugel. Hab. Schrift, Jena, 1875.

TAIT. On Ångström's Method for the Conductivity in Bars. Edinburgh, Proc. R. Soc., 8, 1875.

PURSER. An Application of Elliptic Functions to a Problem in the Distribution of Heat in a Rectangular Lamina. Mess. Math., Cambridge, 6, 1877.

AYRTON AND PERRY. On the Heat Conductivity of Stone. Phil. Mag., London (Ser. 5), 5, 1878.

KELVIN. Problems relating to Underground Temperature. Phil Mag., London (Ser. 5), 5, 1878.

LODGE. On a Method of measuring the Absolute Thermal Conductivity of Crystals, and other rare Substances. Phil. Mag., London (Ser. 5), 5, 1878.

MATHIEU. Étude des solutions simples des équations aux différences partielles de la physique mathématique. J. math., Paris (Sér. 3), 5, 1879.

TAIT. Thermal and Electric Conductivity. Edinburgh, Trans. R. Soc., 28, 1879.

BOUSSINESQ. Sur les problèmes des températures stationaires. J. math., Paris (Sér. 3), 6, 1880.

KIRCHHOFF UND HANSEMANN. Über die Leitungsfähigkeit des Eisens für die Wärme. Ann. Physik, Leipzig (N. Folge), 9, 1880.

NIVEN. On Heat Conduction in Ellipsoids of Revolution. London, Phil. Trans. R. Soc., 171, 1880.

WEBER, H. F. Die Beziehungen zwischen dem Wärmeleitungsvermögen und dem elektrischen Leitungsvermögen der Metalle. Berlin, SitzBer. Ak. Wiss., 1880.

BETTI. Sopra la propagazione del calore. Chelini Coll. mat., Milano, 1881.

KIRCHHOFF UND HANSEMANN. Über die Leitungsfähigkeiten der Metalle für Wärme und Elektricität. Ann. Physik, Leipzig (N. Folge), 13, 1881.

LORBERG. Über Wärmeleitung in einem System von Cylindern. Ann. Physik, Leipzig (N. Folge), 14, 1881.

LORENZ. Über das Leitungsvermögen der Metalle für Wärme und Elektricität. Ann. Physik, Leipzig (N. Folge), 13, 1881.

MOLLISON. Note on Conduction of Heat. Mess. Math., Cambridge, 10, 1881.

TAIT. Note on Thermal Conductivity and on the Effects of Temperature Changes of Specific Heat and Conductivity on the Propagation of Plane Heat Waves. Phil. Mag., London (Ser. 5), 12, 1881.

RESAL. Commentaire à la théorie analytique de la chaleur de Fourier. J. math., Paris (Sér. 3), 8, 1882.

BOTTOMLEY. On the permanent Temperature of Conductors through which an Electric Current is passing, and on Surface Conductivity or Emissivity. With a note by Sir William Thomson. London, Proc. R. Soc., 37, 1884.

FUDZISAWA. Über eine in die Wärmeleitungstheorie auftretende nach den Wurzeln einer transcendenten Gleichung fortschreitende unendliche Reihe. Diss. Strassburg, 1886.

BELTRAMI. Intorno ad alcuni problemi di propagazione del calore. Bologna, Mem. Acc. sci. (Ser. 4), 8, 1887. (See also Nuovo Cimento, Pisa (Ser. 3), 23-25, 1888.)

HARNACK. Zur Theorie der Wärmeleitung in festen Körpern. Zs. Math., Leipzig, 32, 1887.

POINCARÉ. Sur la théorie de la chaleur. Paris, C.R. Acad. sci., 104, 1887.

WOODWARD. On the free Cooling of a Homogeneous Sphere of Initial Uniform Temperature in a Medium which maintains a Constant Surface Temperature. Ann. Math., Camb. Mass., 3, 1887.

POINCARÉ. Sur la théorie de la chaleur. Paris, C.R. Acad. sci., 107, 1888.

WOODWARD. (i) On the Conditioned Cooling and the Contraction of a Homogeneous Sphere. Ann. Math., Camb. Mass., 4, 1888.

(ii) On the Diffusion of Heat in a Homogeneous Rectangular Mass, with special reference to Bars used as Standards of Length. Ann. Math., Camb. Mass., 4, 1888.

HOBSON. (i) On a Radiation Problem. Cambridge, Proc. Phil. Soc., 6, 1889.
(ii) Synthetical Solutions in the Conduction of Heat. London, Proc. Math. Soc., 19, 1889.

LEES. On the Law of Cooling and its Bearing on certain Equations in the Analytical Theory of Heat. Phil. Mag., London (Ser. 5), 28, 1889.

CHWOLSON. (i) Über einem Fall von variabler Temperaturvertheilung in einem Stabe. Exner's Repertorium, 26, 1890.
(ii) Über die Abhängigkeit der Wärmeleitungsfähigkeit von der Temperatur. St. Petersburg, Mém. Acad. Sci., 37, 1890.

STEFAN. Über einige Probleme der Theorie der Wärmeleitung. Wien, SitzBer. Ak. Wiss., 98, 1890.

BRYAN. An Application of the Method of Images to the Conduction of Heat. London, Proc. Math. Soc., 22, 1891.

CHWOLSON. Über die Vertheilung der Wärme in einer einseitig bestrahlten schwarzen Kugel. St. Petersburg, Mém. Acad. Sci., 38, 1891.

HAGSTRÖM. Vergleichende Untersuchungen über die Methode von Ångström und Neumann zur Bestimmung der Wärmeleitung der Körper. Stockholm, Vet. Ak. Öfvers, 48, 1891.

LINDE. (i) Methode zur Bestimmung der Wärmeleitungsvermögen in einer Kugel. Exner's Repertorium, 27, 1891.
(ii) Über die Temperaturbestimmung eines Drahtes wenn durch denselben ein galvanischer Strom durchfliesst. Exner's Repertorium, 27, 1891.

APPELL. Sur l'équation $\frac{\partial^2 z}{\partial x^2}-\frac{\partial z}{\partial y}=0$, et la théorie de la chaleur. J. math., Paris (Sér. 4), 8, 1892.

BRILL. A property of the Equation of the Conduction of Heat. Mess. Math., Cambridge, 21, 1892.

BRYAN. Note on a Problem in the Linear Conduction of Heat. Cambridge, Proc. Phil. Soc., 7, 1892.

KOBALD. Über einige particuläre Lösungen der Differentialgleichung für die Wärmeleitung in einem Kreiscylinder und deren Anwendungen. Wien, SitzBer Ak. Wiss., 102, 1893.

LEES. On the Thermal Conductivity of Crystals and other Bad Conductors. London, Phil. Trans. R. Soc., 183, 1893.

WEBER, H. Über den Temperatur-Ausgleich zwischen zwei sich berührenden heterogenen Körpern. Göttingen, Nachr. Ges. Wiss., 1893.

CHWOLSON. Zwei Wärmeleitungs-probleme. Ann. Physik, Leipzig (N. Folge), 51, 1894.

CZERMAK. Über die Temperaturvertheilung eines dünnen Drahtes, der von einem constanten Strome durchflossen wird. Wien, SitzBer. Ak. Wiss., 103, 1894.

POINCARÉ. Sur les équations de la physique mathématique. Palermo, Rend. Circ. mat., 8, 1894.

SOMMERFELD. Zur analytischen Theorie der Wärmeleitung. Math. Ann., Leipzig, 45, 1894.

LACOUR. Sur l'équation de la chaleur. Ann. Fac. Sci., Toulouse, 9, 1895.

PERRY. On the Age of the Earth. Nature, London, 51, 1895.

SOMIGLIANA. Sul problema della temperatura nell' ellissoïde. Ann. mat., Milano (Ser. 2), 24, 1896.

VOIGT. Eine neue Methode zur Untersuchung der Wärmeleitung in Krystallen. Göttingen, Nachr. Ges. Wiss., 1896.

BOULANGER. Sur l'équation de la propagation de la chaleur. Paris, Bul. soc. math., 25, 1897.

LAURICELLA. Sulle temperature stazionarie. Palermo, Rend. Circ. mat., 11, 1897.

LE ROY. Sur l'intégration des équations de la chaleur. Ann. sci. Éc. norm., Paris (Sér. 3), 14, 1897. (Also, 15, 1898.)

VOIGT. Bestimmung relativer Wärmeleitungsfähigkeiten nach der isothermen Methode. Göttingen, Nachr. Ges. Wiss., 1897.

ASCOLI. Sulla determinazione della temperatura e dei coefficienti di conduttività interna ed esterna di un conduttore. Nuovo Cimento, Pisa (Ser. 4), 7, 1898.

CARSLAW. Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications. London, Proc. Math. Soc., 30, 1898.

LAURICELLA. Sulla propagazione del calore. Torino, Atti Acc. sci., 33, 1898.

LEES. On the Thermal Conductivities of Single and Mixed Solids and their Variations with Temperature. London, Phil. Trans. R. Soc., 191, 1898.

SCHULZE. Über eine Methode zur Bestimmung der Wärmeleitung fester Körper. Ann. Physik, Leipzig (N. Folge), 66, 1898.

STEKLOFF. (i) Sur le problème de refroidissement d'une barre hétérogène. Paris, C.R. Acad. sci., 126, 1898.

(ii) Sur un problème de la théorie analytique de la chaleur. Paris, C.R. Acad. sci., 126, 1898.

STRANEO. (i) Sulla temperatura di un conduttore lineare bimetallico. Roma, Rend. Acc. Lincei (Ser. 5), 7, 1898.

(ii) Sulla determinazione simultanea delle conducibilitá termiche ed electrice dei metalli a differenti temperature. Roma, Rend. Acc. Lincei (Ser. 5), 7, 1898.

KOHLRAUSCH. Über den stationären Temperaturzustand eines von einem elektrischen Strome erwärmten Leiters. Berlin, SitzBer. Ak. Wiss., 1899.

PIERCE AND WILLSON. On the Thermal Conductivities of certain Poor Conductors. Boston, Mass., Proc. Amer. Acad. Arts Sci., 34, 1899.

VOIGT. Über ein von Herrn Fr. Kohlrausch aufgestelltes Problem der Wärmelehre. Göttingen, Nachr. Ges. Wiss., 1899.

BOUSSINESQ. (i) Réduction de certains problèmes d'échauffement ou de refroidissement par rayonnement au cas plus simple de l'échauffement ou du refroidissement des mêmes corps par contact; échauffement d'un mur d'épaisseur indéfinie. Paris, C.R. Acad. sci., 130, 1900.

(ii) Problème du refroidissement de la croûte terrestre, traité au même point de vue que l'a fait Fourier, mais par une méthode d'intégration beaucoup plus simple. Paris, C.R. Acad. sci., 130, 1900.

(iii) Problème du refroidissement d'un mur par rayonnement ramené au cas plus simple où le refroidissement aurait lieu par contact. Paris, C.R. Acad. sci., 130, 1900.

(iv) Échauffement permanent mais inégal par rayonnement, d'un mur d'épaisseur indéfinie, ramené au cas d'un échauffement analogue par contact. Paris, C.R. Acad. sci., 131, 1900.

(v) Problème de l'échauffement permanent d'une sphère par rayonnement, ramené au problème plus simple de l'échauffement de la même sphère par contact. Paris, C.R. Acad. sci., 131, 1900.

COTTON. Mouvement de la chaleur sur la surface d'un tétraèdre dont les arêtes opposées sont égales. Ann. Fac. Sci., Toulouse (Sér. 2), 2, 1900.

GRUNEISEN. Über die Bestimmung des metallischen Wärmeleitvermögens und über sein Verhältniss zur elektrischen Leitfähigkeit. Ann. Physik, Leipzig (4. Folge), 3, 1900.

HALL. Concerning Thermal Conductivity in Iron. Physic. Rev., Ithaca, N.Y., 10, 1900.

KOHLRAUSCH. Über den stationären Temperaturzustand eines elektrischgeheisten Leiters. Ann. Physik, Leipzig (4. Folge), 1, 1900.

PEIRCE. On the Thermal Conductivity of Vulcanite. Phil. Mag., London (Ser. 5), 49, 1900.

PICARD. (i) Sur l'équilibre calorifique d'une surface fermée rayonnant au dehors. Paris, C.R. Acad. sci., 130, 1900.

(ii) Sur quelques problèmes relatifs à l'équation $\Delta u = \kappa^2 u$. Bull. Sci. Math. de France, 28, 1900.

STEKLOFF. Le problème des températures stationaires. Paris, C.R. Acad. sci., 131, 1900.

BOUSSINESQ. Problème de la dissipation en tous sens de la chaleur dans un mur épais à face rayonnement. Paris, C.R. Acad. sci., 133, 1901.

DOUGALL. Note on the Application of Complex Integration to the Equation of Conduction of Heat, with a special application to Dr. Peddie's Problem. Edinburgh, Proc. Math. Soc., 19, 1901.

PEDDIE. Note on the Cooling of a Sphere in a Mass of well-stirred Liquid. Edinburgh, Proc. Math. Soc., 19, 1901.

STEKLOFF. Problème du refroidissement d'une barre hétérogène. Ann. Fac. Sci., Toulouse (Sér. 2), 3, 1901.

CARSLAW. A Problem in Conduction of Heat. Phil. Mag., London (Ser. 6), 4, 1902.

CESÀRO. (i) Sur un problème de propagazione de la chaleur. Bruxelles, Bul. Acad. roy., 1902.

(ii) Intorno ad una limitazione di constanti nella teoria analitica del calore. Napoli, Rend. Acc. sci. (Ser. 3), 8, 1902.

LAURICELLA. Sull 'integrazione delle equazioni della propagazione del calore. Roma, Mem. Soc. XL. (Ser. 3), 12, 1902.

MACKENZIE. On some Equations pertaining to the Propagation of Heat in an Infinite Medium. Philadelphia, Pa., Proc. Amer. Phil. Soc., 41, 1902.

PECK. The Steady Temperatures of a Thin Rod. Phil. Mag., London (Ser. 6), 4, 1902.

SCHAUFELBERGER. Wärmeleitungsfähigkeit des Kupfers, aus dem stazionären und variablen Temperaturzustand bestimmt, und Wärmefluss in einer durch Kühlwasser bespülten Endfläche eines Wärmeleiters. Ann. Physik, Leipzig (4. Folge), 7, 1902.

STEKLOFF. Sur les problèmes fondamentaux de la physique mathematique. Ann. sci. Éc. norm., Paris (Sér. 3), 19, 1902.

CARSLAW. The Use of Green's Functions in the Mathematical Theory of the Conduction of Heat. Edinburgh, Proc. Math. Soc., 21, 1903.

GIEBE. Über die Bestimmung des Wärmeleitungsvermögen bei tiefen Temperaturen. Berlin, Verh. D. Physik. Ges., 15, 1903.

PRASAD. Constitution of Matter and Analytical Theories of Heat. Göttingen, Abh. Ges. Wiss., 2, 1903.

SOMIGLIANA. Intorno ad un problema di distribuzione termica. Milano, Rend. Ist. lomb. (Ser. 2), 36, 1903.

BOUSSINESQ. Sur la unicité de la solution simple fondamentale et sur l'expression asymptotique des températures dans le problème du refroidissement. Bul. sci. math., Paris (Sér. 2), 28, 1904.

HECHT. F. E. Neumanns Methode zur Bestimmung der Wärmeleitungsfähigkeit schlecht leitender Körper in Kugel- und Würfelform, und ihre Durchführung an Marmor, Glas, etc. Ann. Physik. Leipzig (4. Folge), 14, 1904.

VOLTERRA. Sur les équations différentielles du type parabolique. Paris, C.R. Acad. sci., 139, 1904.

BUHL. Sur l'approximation des fonctions par des polynomes dans ses rapports avec la théorie des équations aux dérivées partielles; application au problème de l'état initial in physique mathématique. Paris, C.R. Acad. sci., 140, 1905.

GLAGE. F. E. Neumanns Methode zur Bestimmung der Wärmeleitfähigkeit gut leitender Körper in Stab- und Ringform, und ihre Durchführung an Eisen, Stahl, etc. Ann. Physik, Leipzig (4. Folge), 18, 1905.

LEES. The Effects of Temperature and Pressure on the Thermal Conductivities of Bodies. Part I. The Effect of Temperature on the Thermal Conductivity of some Electrical Insulators. London, Phil. Trans. R. Soc., **204**, 1905.

NIVEN. On a Method of finding the Conductivity for Heat. London, Proc. R. Soc., **76**, 1905.

ZAREMBA. Solution générale du problème de Fourier. Kraków, Bull. Intern. Acad., 1905.

PICARD. (i) Sur quelques problèmes de physique mathématique se rattachant à l'équation de M. Fredholm. Paris, C.R. Acad sci., **142**, 1906.

(ii) Sur quelques applications de l'équation fonctionelle de M. Fredholm. Palermo, Rend. Circ. mat., **22**, 1906.

THOMA. Wärmeleitungsproblem bei wellig begrenzten Oberfläche und dessen Anwendung auf Tunnelbauten. Diss. Freiburg u. Br., 1906.

HOLMGREN. Sur l'équation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$. Paris, C.R. Acad. sci., **145**, 1907. (See also **146**, 1908.)

KÖNIGSBERGER UND DISCH. Bestimmung der Veränderlichkeit des Koefficienten der Differentialgleichung von Fourier und experimentelle Anwendung auf Wärmeleitung von Isolatoren. Ann. Physik, Leipzig (4. Folge), **23**, 1907.

LEVI. Sull' equazione del calore. Roma, Rend. Acc. Lincei (Ser. 5), **16**, 1907.

MYLLER-LEBEDEFF. Die Theorie der Integralgleichungen in Anwendung auf einige Reihenentwickelungen (Wärmeleitung in einem Kreiscylinder). Math. Ann., Leipzig, **64**, 1907.

PICCIATTI. Sull' equazione della propagazione di calore in un filo. Roma, Rend. Acc. Lincei (Ser. 5), **16**, 1907.

STEKLOFF. Un problème d'analyse intimement lié au problème du refroidissement d'une barre hétérogène. Paris, C.R. Acad. sci., **144**, 1907.

TEDONE. Sul problema dell' equilibrio delle temperature in un ellissoide a tre assi disuguali. Palermo, Rend. Circ. mat., **24**, 1907.

EBELING. Über den Temperaturverlauf in wechselstromdurchflossenen Drähten. Ann. Physik, Leipzig (4. Folge), **27**, 1908.

HOLMGREN. Sur l'équation de la propagation de la chaleur. Arkiv. Matem., Stockholm, **4**, 1908.

LAURICELLA. Applicazione della teoria di Fredholm al problema del raffreddamento dei corpi. Ann. mat., Milano (Ser. 3), **14**, 1908.

LEES. The Effects of Temperature and Pressure on the Thermal Conductivities of Solids. Part II. The Effects of low Temperatures on the Thermal and Electrical Conductivities of certain approximately pure Metals and Alloys. London, Phil. Trans. R. Soc., 208, 1908. (Also Proc. R. Soc., 80.)

LEVI. (i) Sul problema di Fourier. Torino, Atti Acc. sci., 43, 1908.

(ii) Sur l'équation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$. Paris, C.R. Acad. sci., 146, 1908.

MYLLER-LEBEDEFF. Über die Anwendung der Integralgleichungen in einer parabolischen Randwertaufgabe. Math. Ann., Leipzig, 66, 1908.

LAROSE. (i) Sur les solutions particulières de l'équation $\frac{\partial^2 \phi}{\partial^2 x} - \frac{\partial \phi}{\partial t} = 0$. Paris, C.R. Acad. sci., 148, 1909.

(ii) Sur le problème de l'armille de Fourier. Paris, C.R. Acad. sci., 148, 1909.

(iii) Sur le problème de l'armille avec deux ruptures. Paris, C.R. Acad. sci., 148, 1909.

PICARD. (i) Quelques remarques sur les équations intégrales de première espèce et sur certains problèmes de physique mathématique. Paris, C.R. Acad. sci., 148, 1909.

(ii) Sur une équation aux dérivées partielles du second ordre, relative à une surface fermée, correspondant à un équilibre calorifique. Ann sci. Éc. norm., Paris, 26, 1909.

CARSLAW. The Green's Function for a Wedge of any Angle and other Problems in the Conduction of Heat. London, Proc. Math. Soc. (Ser. 2), 8, 1910.

LEES. On the Shapes of the Isogeotherms under Mountain Ranges in radioactive Districts. London, Proc. R. Soc., 83, 1910.

PICARD. Sur un théorème général relatif aux équations intégrales de première espèce et sur quelques problèmes de physique mathématique. Palermo, Rend. Circ. mat., 29, 1910.

MARCOLONGO. Sull' equazione della propagazione del calore nei corpi cristallizzati. Napoli, Rend. Acc. sc. (Ser. 3), 17, 1911.

RAYLEIGH. Problems in the Conduction of Heat. Phil. Mag., London (Ser. 6), 22, 1911.

SUCHÝ. Wärmestrahlung und Wärmeleitung. Ann. Physik, Leipzig (4. Folge), 36, 1911.

CARSLAW. A problem in the linear flow of heat discussed from the point of view of the theory of integral equations. Edinburgh, Proc. Math. Soc., 30, 1912.

SILLA. Sulla propagazione del calore. Roma, Rend. Acc. Lincei (Ser. 5), 21, 1912.

VOLTERRA. Sulla temperatura nell' intorno delle montagne. Nuovo Cimento, Pisa (Ser. 6), 4, 1912.

WEBER. Über den Eindeutigkeitsbeweis in der Theorie der Wärmeleitung. Heinrich Weber Festschrift, 1912.

MILANKOWITSCH. Über ein Problem der Wärmeleitung und dessen Anwendung auf die Theorie des solaren Klimas. Zs. Math., Leipzig, 62, 1913.

PICARD. Application de la théorie des équations intégrales à certains problèmes de la théorie analytique de la chaleur dans l'hypothèse d'un saut brusque de température à la surface de séparation des corps en contact. Paris, C.R. Acad. sci., 156, 1913.

SOMIGLIANA (E VERCELLI). Sulla previsione matematica della temperatura nei grandi trafori alpini. Torino, Mem. Acc. sci. (Ser. 2), 63, 1913. (Also Vercelli, Torino, Atti Acc. sc., 48, 1913.)

STEKLOFF. Sur certaines questions d'analyse qui se rattachent à plusieurs problèmes de la physique mathématique. St. Petersburg, Mém. Acad. sci. (Sér. 8), 31, 1913.

VERCELLI. Sulla determinazione dei coefficienti di conduttività termica mediante il raffreddamento di sfere. Nuovo Cimento, Pisa (Ser. 6), 6, 1913.

WEINREICH. Über den Temperaturverlauf in stromdurchflossenen Drähten, besonders im Fall von Wechselstrom. Zs. Math., Leipzig, 63, 1914.

BOUSSINESQ. (i) Calcul correct de l'influence de l'inégalité climatérique sur la vitesse d'accroissement des températures terrestres avec la profondeur sous le sol. Bul. sci. math., Paris (Sér. 2), 39, 1915.

(ii) Sur le problème du refroidissement de la croûte terrestre considéré à la manière et suivant les idées de Fourier. Bul. sci. math., Paris (Sér. 2), 39, 1915.

DATTA. On the non-stationary state of heat in an ellipsoid. Bull. Calcutta Math. Soc., 8, 1915.

LAUDIEN. Entwicklung willkürlicher Funktionen bei einem thermoelastischen Problem. J. Math., Berlin, 148, 1918.

AICHI. (i) On Picard's solution of $\Delta\theta = k^2\theta$. Proc. Phys.-math. Soc., Japan (Ser. 3), 1, 1919.

(ii) Heat Distribution on a radiating plane. Proc. Phys.-math. Soc., Japan (Ser. 3), 1, 1919.

BROMWICH. Examples of operational methods of solving problems in the conduction of heat. Phil. Mag., London (Ser. 6), 37, 1919.

McLEOD. On the lags of thermometers with spherical and cylindrical bulbs in a medium whose temperature is changing at a constant rate. Phil. Mag., London (Ser. 6), 37, 1919.

AICHI. (i) Heat Distribution on a radiating plane, and especially when the boundary is circular. Proc. Phys.-math. Soc., Japan (Ser. 3), 2, No. 2 and No. 6, 1920.

(ii) On the two-dimensional convection of heat by the uniform current of steam. Proc. Phys.-math. Soc., Japan (Ser. 3), 2, No. 7, 1920.

CARSLAW. On Bromwich's method of solving problems in the conduction of heat. Phil. Mag., London (Ser. 6), 39, 1920.

HAYASHI. On Picard's solution of $\Delta u = k^2 u$. Science Reports of the Tokuhu Imperial University, Sendai, Japan (Ser. 1), 9, 1920.

BROMWICH. Symbolical methods in the theory of conduction of heat. Cambridge, Proc. Phil. Soc., 20, 1921.

CARSLAW. The cooling of a solid sphere with a concentric core of a different material. Cambridge, Proc. Phil. Soc., 20, 1921.

OWEN. On the lag of a thermometer in a medium whose temperature is a linear function of the time. London, Proc. Math. Soc. (Ser. 2), 19, 1921.

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PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW.